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An $\alpha^2(Z\alpha)^5m$ Contribution to the Hydrogen
Lamb Shift
from Virtual Light by Light Scattering

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Abstract

The radiative correction to the Lamb shift of order $\alpha^2(Z\alpha)^5m$ induced by the light by light scattering insertion in external photons is obtained. The new contribution turns out to be equal to $-0.122(2)\alpha^2(Z\alpha)^5/(\pi n^3)(m_r/m)^3m$. Combining this contribution with our previous results we obtain the complete correction of order $\alpha^2(Z\alpha)^5m$ induced by all diagrams with closed electron loops. This correction is 37.3(1) kHz and 4.67(1) kHz for the $1S$ - and $2S$ -states in hydrogen, respectively.

1 INTRODUCTION

Recently we started a calculation of all contributions to the Lamb shift of order $\alpha^2(Z\alpha)^5m$. It was shown that there exist six gauge invariant sets of diagrams, which produce such corrections. All these diagrams may be obtained by different dressings from the skeleton diagram which contains two external photons attached to the electron line. Contributions induced by polarization operator insertions in external photons, by simultaneous insertion of a radiative photon in electron line and a one-loop polarization operator in the external photons and by polarization operator insertions in radiative photons have been calculated in [1, 2, 3].

We present below a calculation of the $\alpha^2(Z\alpha)^5m$ contribution to the Lamb shift in hydrogenlike ions, induced by penultimate gauge invariant set of graphs. These are the graphs containing one-loop light by light scattering insertions between two external photon lines (see Fig.1).

The contribution to the Lamb shift, produced by the diagrams in Fig.1 is given by the expression (see, e.g. [1])

$$\Delta E = \frac{\alpha^2(Z\alpha)^5}{\pi n^3} m \left(\frac{m_r}{m}\right)^3 \frac{1}{\pi^2} \int \frac{d^4 q}{i} \frac{1}{q^4} \int \frac{d^3 \mathbf{k}}{4\pi k^4} (\mathcal{A}_{\alpha\beta} + \mathcal{B}_{\alpha\beta}) \mathcal{S}_{\alpha\beta 00}, \quad (1)$$

where q_μ is the four-momentum of the upper photon lines, $k_\mu = (0, \mathbf{k})$ is the spatial momentum of external photons, $\mathcal{A}_{\alpha\beta}$ and $\mathcal{B}_{\alpha\beta}$ are electron-line factors, and $\mathcal{S}_{\alpha\beta 00}$ is the light by light scattering tensor¹.

2 SIMPLIFICATION OF THE EXPRESSION FOR THE ENERGY SHIFT

We begin the calculation with a simplification of the integrand in eq.(1). The general expression for the light by light scattering tensor is

$$\mathcal{S}_{\alpha\beta 00} = \int \frac{d^4 p}{\pi^2 i} (2L_{\alpha\beta 00} + C_{\alpha\beta 00}) \equiv \int \frac{d^4 p}{\pi^2 i} S_{\alpha\beta 00}, \quad (2)$$

where

¹Really $\mathcal{S}_{\alpha\beta 00}$ is defined as one fourth of the light by light scattering tensor; see normalization conventions for this tensor as well as for the electron factor in the next section.

$$L_{\alpha\beta 00} = -\frac{1}{4D_1^2 D_2 D_3} \text{Tr}\{\gamma_0(\hat{p} - \hat{k} + m)\gamma_0(\hat{p} + m)\gamma_\beta(\hat{p} - \hat{q} + m)\gamma_\alpha(\hat{p} + m)\}, \quad (3)$$

corresponds to each of the ladder diagrams *a* and *b* in Fig.1, and

$$C_{\alpha\beta 00} = -\frac{1}{4D_1 D_2 D_3 D_4} \text{Tr}\{\gamma_0(\hat{p} - \hat{k} + m)\gamma_\beta(\hat{p} - \hat{k} - \hat{q} + m)\gamma_0(\hat{p} - \hat{q} + m)\gamma_\alpha(\hat{p} + m)\} \quad (4)$$

corresponds to the crossed diagram *c* in Fig.1. Factors in the denominators above are defined as follows

$$D_1 = p^2 - m^2, D_2 = (p - k)^2 - m^2, D_3 = (p - q)^2 - m^2, D_4 = (p - k - q)^2 - m^2. \quad (5)$$

The explicit expression for the electron factor in eq.(1) is the sum of two terms, where

$$\begin{aligned} \mathcal{A}_{\alpha\beta} &= \frac{\text{Tr}\{(\gamma_0 + 1)\gamma_\alpha(\gamma_0 m + \hat{q} + m)\gamma_\beta\}}{4(q^2 + 2mq_0)} m^2 \\ &= \frac{-g_{\alpha\beta}q_0 + 2g_{\alpha 0}g_{\beta 0}m + g_{\alpha 0}q_\beta + q_\alpha g_{\beta 0}}{q^2 + 2mq_0} m^2 \equiv \frac{A_{\alpha\beta}}{q^2 + 2mq_0} \end{aligned} \quad (6)$$

corresponds to the diagram with nonintersecting upper photon lines and

$$\begin{aligned} \mathcal{B}_{\alpha\beta} &= \frac{\text{Tr}\{(\gamma_0 + 1)\gamma_\beta(\gamma_0 m - \hat{q} + m)\gamma_\alpha\}}{4(q^2 - 2mq_0)} m^2 \\ &= \frac{g_{\alpha\beta}q_0 + 2g_{\alpha 0}g_{\beta 0}m - g_{\alpha 0}q_\beta - q_\alpha g_{\beta 0}}{q^2 - 2mq_0} m^2 \equiv \frac{B_{\alpha\beta}}{q^2 - 2mq_0} \end{aligned} \quad (7)$$

corresponds to the diagram with crossed upper photon lines ².

Note that $\mathcal{A}_{\alpha\beta}(q) = \mathcal{B}_{\alpha\beta}(-q)$ and both $\mathcal{A}_{\alpha\beta}$ and $\mathcal{B}_{\alpha\beta}$ are symmetric relative to permutation of indices. Hence, only the even in q and symmetric under permutation of indices α and β part of the light by light scattering tensor is relevant for calculations below. Due to gauge invariance terms in the light by light scattering tensor which are proportional to vectors q_α and

²We have included an additional factor of m^2 in the definition of the electron line factor simply to preserve the appearance of the expression for the energy shift in eq.(1) even after transition to the dimensionless momenta.

q_β are also irrelevant for calculation of the contribution to the Lamb shift and we omit such terms below.

Since $\mathcal{A}_{\alpha\beta}(q) = \mathcal{B}_{\alpha\beta}(-q)$ the expression for the energy shift may be presented in the following form

$$\begin{aligned}\Delta E &= \frac{\alpha^2(Z\alpha)^5}{\pi n^3} m \left(\frac{m_r}{m}\right)^3 \frac{1}{\pi^2} \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^4} \int \frac{d^3 \mathbf{k}}{4\pi \mathbf{k}^4} \mathcal{A}_{\alpha\beta}(q) [\tilde{\mathcal{S}}_{\alpha\beta 00}(q) + \tilde{\mathcal{S}}_{\alpha\beta 00}(-q)] \\ &= \frac{\alpha^2(Z\alpha)^5}{\pi n^3} m \left(\frac{m_r}{m}\right)^3 \frac{1}{\pi^2} \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^4} \int \frac{d^3 \mathbf{k}}{4\pi \mathbf{k}^4} \times \\ &\quad \int \frac{d^4 p}{\pi^2 i} \mathcal{A}_{\alpha\beta}(q) [\tilde{\mathcal{S}}_{\alpha\beta 00}(q, k, p) + \tilde{\mathcal{S}}_{\alpha\beta 00}(-q, k, p)],\end{aligned}\tag{8}$$

where $\tilde{\mathcal{S}}_{\alpha\beta 00}$ is the integrand of the light by light scattering tensor symmetrized in α and β .

Both k and p are dummy integration momenta and we use substitutions $k \rightarrow -k$ and $p \rightarrow -p$ while calculating the second product in the integrand in eq.(8). After these substitutions denominators in the integral representation for the light by light scattering tensor with external momenta of opposite sign return to the original form (see eq.(3) and eq.(4)). Moreover, it is easy to see that

$$\tilde{\mathcal{S}}_{\alpha\beta 00}(-q, -k, -p; m) = \tilde{\mathcal{S}}_{\alpha\beta 00}(q, k, p; -m),\tag{9}$$

if one recollects that there exists an additional argument m in the light by light scattering tensor and takes into account explicit expressions in eq.(3) and eq.(4). Traces in the numerators on the right hand sides in these equations convert these numerators into even polynomials of m and, hence, the expression for the energy splitting in eq.(8) reduces to

$$\begin{aligned}\Delta E &= \\ &\frac{\alpha^2(Z\alpha)^5}{\pi n^3} m \left(\frac{m_r}{m}\right)^3 \frac{2}{\pi^2} \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^4} \int \frac{d^3 \mathbf{k}}{4\pi \mathbf{k}^4} \int \frac{d^4 p}{\pi^2 i} \mathcal{A}_{\alpha\beta}(q) \tilde{\mathcal{S}}_{\alpha\beta 00}(q, k, p; m).\end{aligned}\tag{10}$$

3 CALCULATION OF THE LIGHT BY LIGHT SCATTERING TENSOR

3.1 LADDER DIAGRAM

The explicit expression for the tensor in eq.(3) symmetrized over indices α and β , has the form

$$\begin{aligned} \tilde{L}_{\alpha\beta 00} = & -\frac{1}{D_1^2 D_2 D_3} \{ -(kp)p^2 g_{\alpha\beta} + 2(kp)(pq)g_{\alpha\beta} + 4(kp)p_\alpha p_\beta \\ & + (kp)g_{\alpha\beta}m^2 - (kq)p^2 g_{\alpha\beta} + (kq)g_{\alpha\beta}m^2 - k_\alpha p^2 p_\beta + k_\alpha p_\beta m^2 - k_\beta p^2 p_\alpha + k_\beta p_\alpha m^2 \\ & + p^4 g_{\alpha\beta} - p^2(pq)g_{\alpha\beta} - 2p^2 p_0^2 g_{\alpha\beta} - 2p^2 p_0 p_\alpha g_{\beta 0} - 2p^2 p_0 p_\beta g_{\alpha 0} - 2p^2 p_0 q_0 g_{\alpha\beta} - 2p^2 p_\alpha p_\beta \\ & - 2p^2 g_{\alpha\beta}m^2 + 4(pq)p_0^2 g_{\alpha\beta} + (pq)g_{\alpha\beta}m^2 + 8p_0^2 p_\alpha p_\beta + 2p_0^2 g_{\alpha\beta}m^2 + 2p_0 p_\alpha g_{\beta 0}m^2 \\ & + 2p_0 p_\beta g_{\alpha 0}m^2 + 2p_0 q_0 g_{\alpha\beta}m^2 + 2p_\alpha p_\beta m^2 + g_{\alpha\beta}m^4 \}. \end{aligned} \quad (11)$$

Next we combine denominators with the help of the Feynman parameters x and u

$$\begin{aligned} (1-u)D_1 + u[xD_2 + (1-x)D_3] &= p'^2 - \Delta, \\ p' &= p - kux - qu(1-x), \\ \Delta &= -k^2 xu(1-ux) - q^2 u(1-x)[1-u(1-x)] + 2(kq)u^2 x(1-x) + m^2, \\ \frac{1}{D_1^2 D_2 D_3} &= 6 \int_0^1 dx \int_0^1 du u(1-u) \frac{1}{(p'^2 - \Delta)^4}. \end{aligned} \quad (12)$$

Thus we obtain

$$\begin{aligned} \tilde{\mathcal{L}}_{\alpha\beta 00} = & \\ -6 \int_0^1 dx \int_0^1 du u(1-u) \int \frac{d^4 p'}{\pi^2 i} \{ & \frac{1}{3} p'^4 \tilde{L}_{\alpha\beta 00}^{(0)} + \frac{1}{2} p'^2 \tilde{L}_{\alpha\beta 00}^{(1)} + \tilde{L}_{\alpha\beta 00}^{(2)} \} \frac{1}{(p'^2 - \Delta)^4}, \end{aligned} \quad (13)$$

where

$$\tilde{L}_{\alpha\beta 00}^{(0)} = -g_{\alpha 0} g_{\beta 0} + g_{\alpha\beta}, \quad (14)$$

$$\begin{aligned} \tilde{L}_{\alpha\beta 00}^{(1)} = & -2k^2 g_{\alpha 0} g_{\beta 0} (ux)^2 + 4k^2 g_{\alpha\beta} (ux)^2 - k^2 g_{\alpha\beta} (ux) \\ & - 4(kq)g_{\alpha 0} g_{\beta 0} (ux)[u(1-x)] + 8(kq)g_{\alpha\beta} (ux)[u(1-x)] - (kq)g_{\alpha\beta} (ux) \\ & - (kq)g_{\alpha\beta} [u(1-x)] - (kq)g_{\alpha\beta} - 4k_\alpha k_\beta (ux)^2 - 2q^2 g_{\alpha 0} g_{\beta 0} [u(1-x)]^2 \\ & + 4q^2 g_{\alpha\beta} [u(1-x)]^2 - q^2 g_{\alpha\beta} [u(1-x)] - 4q_0^2 g_{\alpha\beta} [u(1-x)]^2 \end{aligned} \quad (15)$$

$$-2q_0^2 g_{\alpha\beta} [u(1-x)] + 2g_{\alpha 0} g_{\beta 0} m^2 - 2g_{\alpha\beta} m^2,$$

$$\begin{aligned} \tilde{L}_{\alpha\beta 00}^{(2)} = & k^4 g_{\alpha\beta} (ux)^4 - k^4 g_{\alpha\beta} (ux)^3 + 4k^2 (kq) g_{\alpha\beta} (ux)^3 [u(1-x)] \quad (16) \\ & - k^2 (kq) g_{\alpha\beta} (ux)^3 - 3k^2 (kq) g_{\alpha\beta} (ux)^2 [u(1-x)] + k^2 (kq) g_{\alpha\beta} (ux)^2 \\ & - 2k^2 k_\alpha k_\beta (ux)^4 + 2k^2 k_\alpha k_\beta (ux)^3 - 2k^2 k_\alpha q_0 g_{\beta 0} (ux)^3 [u(1-x)] \\ & - 2k^2 k_\beta q_0 g_{\alpha 0} (ux)^3 [u(1-x)] + 2k^2 q^2 g_{\alpha\beta} (ux)^2 [u(1-x)]^2 - k^2 q^2 g_{\alpha\beta} (ux)^2 [u(1-x)] \\ & - k^2 q^2 g_{\alpha\beta} (ux) [u(1-x)]^2 + 2k^2 q^2 g_{\alpha\beta} (ux) [u(1-x)] - 2k^2 q_0^2 g_{\alpha\beta} (ux)^2 [u(1-x)]^2 \\ & - 2k^2 q_0^2 g_{\alpha\beta} (ux)^2 [u(1-x)] - 2k^2 g_{\alpha\beta} m^2 (ux)^2 + k^2 g_{\alpha\beta} m^2 (ux) \\ & + 4(kq)^2 g_{\alpha\beta} (ux)^2 [u(1-x)]^2 - 2(kq)^2 g_{\alpha\beta} (ux)^2 [u(1-x)] \\ & - 2(kq)^2 g_{\alpha\beta} (ux) [u(1-x)]^2 - 4(kq) k_\alpha k_\beta (ux)^3 [u(1-x)] \\ & - 4(kq) k_\alpha q_0 g_{\beta 0} (ux)^2 [u(1-x)]^2 - 4(kq) k_\beta q_0 g_{\alpha 0} (ux)^2 [u(1-x)]^2 \\ & + 4(kq) q^2 g_{\alpha\beta} (ux) [u(1-x)]^3 - 3(kq) q^2 g_{\alpha\beta} (ux) [u(1-x)]^2 - (kq) q^2 g_{\alpha\beta} [u(1-x)]^3 \\ & + (kq) q^2 g_{\alpha\beta} [u(1-x)]^2 - 4(kq) q_0^2 g_{\alpha\beta} (ux) [u(1-x)]^3 - 4(kq) g_{\alpha\beta} m^2 (ux) [u(1-x)] \\ & + (kq) g_{\alpha\beta} m^2 (ux) + (kq) g_{\alpha\beta} m^2 [u(1-x)] + (kq) g_{\alpha\beta} m^2 \\ & - 2k_\alpha k_\beta q^2 (ux)^2 [u(1-x)]^2 - 2k_\alpha k_\beta q^2 (ux) [u(1-x)]^2 + 8k_\alpha k_\beta q_0^2 (ux)^2 [u(1-x)]^2 \\ & + 2k_\alpha k_\beta m^2 (ux)^2 + 2k_\alpha k_\beta m^2 (ux) - 2k_\alpha q^2 q_0 g_{\beta 0} (ux) [u(1-x)]^3 \\ & + 2k_\alpha q_0 g_{\beta 0} m^2 (ux) [u(1-x)] - 2k_\beta q^2 q_0 g_{\alpha 0} (ux) [u(1-x)]^3 \\ & + 2k_\beta q_0 g_{\alpha 0} m^2 (ux) [u(1-x)] + q^4 g_{\alpha\beta} [u(1-x)]^4 - q^4 g_{\alpha\beta} [u(1-x)]^3 \\ & - 2q^2 q_0^2 g_{\alpha\beta} [u(1-x)]^4 + 2q^2 q_0^2 g_{\alpha\beta} [u(1-x)]^3 - 2q^2 g_{\alpha\beta} m^2 [u(1-x)]^2 \\ & + q^2 g_{\alpha\beta} m^2 [u(1-x)] + 2q_0^2 g_{\alpha\beta} m^2 [u(1-x)]^2 + 2q_0^2 g_{\alpha\beta} m^2 [u(1-x)] + g_{\alpha\beta} m^4. \end{aligned}$$

Next we perform the Wick rotation and integration over the loop momentum p' . The momentum integral diverges logarithmically. It is well known that even this log of the ultraviolet cutoff cancels if one adds the crossed diagram contribution. A finite term breaking gauge invariance remains as the only remnant of the divergence. Since it is still necessary to subtract the value of the light by light scattering tensor at zero momentum in order to

restore gauge invariance we will now perform the subtraction directly in the ladder diagram contribution.

Integration is performed trivially and we obtain

$$\begin{aligned} \tilde{\mathcal{L}}_{\alpha\beta 00} = & \quad (17) \\ -6 \int_0^1 dx \int_0^1 du u(1-u) & \left\{ \frac{1}{3} \left(\log \frac{\Lambda^2}{\Delta} - \frac{11}{6} \right) \tilde{L}_{\alpha\beta 00}^{(0)} - \frac{1}{6\Delta} \tilde{L}_{\alpha\beta 00}^{(1)} + \frac{1}{6\Delta^2} \tilde{L}_{\alpha\beta 00}^{(2)} \right\}. \end{aligned}$$

Next we do the subtraction. It is convenient first to perform auxiliary integration by parts. This gives one a chance to separate a subtraction term connected with the logarithm and to get rid of this log in the integrand simultaneously. We use the identity

$$\begin{aligned} \int_0^1 du u(1-u) \log \frac{\Lambda^2}{\Delta} &= -\frac{(1-u)^2(2u+1)}{6} \log \frac{\Lambda^2}{\Delta} \Big|_0^1 \quad (18) \\ &- \int_0^1 \frac{du}{6\Delta} (1-u)^2(2u+1) \{ -k^2 x(1-2xu) \\ &- q^2(1-x)[1-2u(1-x)] + 4(kq)ux(1-x) \} \\ &= \frac{1}{6} \log \frac{\Lambda^2}{m^2} - \int_0^1 \frac{du}{6\Delta} (1-u)^2(2u+1) \{ -k^2 x(1-2xu) \\ &- q^2(1-x)[1-2u(1-x)] + 4(kq)ux(1-x) \} \end{aligned}$$

to deal with the log on the right hand side in eq.(17).

Subtraction of the logarithmic term at zero momenta is performed simply by omitting the log as well as the constant term after integration by parts.

Some finite terms in eq.(17) also need subtraction. As was mentioned above it is necessary to subtract the value of the ladder contribution to the light by light scattering tensor at zero external momentum. Considering explicit expressions for the functions $\tilde{L}_{\alpha\beta 00}^{(i)}$ in eq.(14) it is easy to see that only terms $2g_{\alpha 0}g_{\beta 0}m^2 - 2g_{\alpha\beta}m^2$ in function $\tilde{L}_{\alpha\beta 00}^{(1)}$ and the term $g_{\alpha\beta}m^4$ in function $\tilde{L}_{\alpha\beta 00}^{(2)}$ need subtraction. Now, subtraction is performed simply by the replacements

$$\begin{aligned} \frac{1}{\Delta} &\rightarrow \frac{1}{\Delta(k, q)} - \frac{1}{\Delta(0, 0)} \quad (19) \\ &= \frac{k^2 xu(1-ux) + q^2 u(1-x)[1-u(1-x)] - 2(kq)u^2 x(1-x)}{m^2 \Delta}, \end{aligned}$$

$$\begin{aligned}
\frac{1}{\Delta^2} &\rightarrow \frac{1}{\Delta^2(k, q)} - \frac{1}{\Delta^2(0, 0)} = \left(\frac{1}{\Delta(k, q)} - \frac{1}{\Delta(0, 0)} \right) \left(\frac{1}{\Delta(k, q)} + \frac{1}{\Delta(0, 0)} \right) \\
&= (2m^2 - k^2xu(1 - ux) - q^2u(1 - x)[1 - u(1 - x)] + 2(kq)u^2x(1 - x)) \\
&\quad \frac{k^2xu(1 - ux) + q^2u(1 - x)[1 - u(1 - x)] - 2(kq)u^2x(1 - x)}{m^4\Delta^2}
\end{aligned}$$

in the denominators of the terms just discussed.

We obtain then the expression for the subtracted ladder diagram contribution

$$\tilde{\mathcal{L}}_{\alpha\beta 00}^{sub} = \quad (20)$$

$$- \int_0^1 dx \int_0^1 du \left\{ \frac{1-u}{3\Delta} \tilde{L}_{\alpha\beta 00}^{(1)sub} + \frac{u(1-u)}{\Delta^2} \tilde{L}_{\alpha\beta 00}^{(2)sub} \right\},$$

where

$$\begin{aligned}
\tilde{L}_{\alpha\beta 00}^{(1)sub} &= -2k^2 g_{\alpha 0} g_{\beta 0} uvx(1 - u) - 6k^2 g_{\alpha 0} g_{\beta 0} uv(ux) \quad (21) \\
&+ 2k^2 g_{\alpha 0} g_{\beta 0} (ux)^2(1 - u) + 6k^2 g_{\alpha 0} g_{\beta 0} u(ux)^2 - k^2 g_{\alpha 0} g_{\beta 0} vx(1 - u) \\
&+ k^2 g_{\alpha 0} g_{\beta 0} x(ux)(1 - u) + 2k^2 g_{\alpha\beta} uvx(1 - u) + 6k^2 g_{\alpha\beta} uv(ux) \\
&- 2k^2 g_{\alpha\beta} (ux)^2(1 - u) - 12k^2 g_{\alpha\beta} u(ux)^2 + 3k^2 g_{\alpha\beta} u(ux) + k^2 g_{\alpha\beta} vx(1 - u) \\
&- k^2 g_{\alpha\beta} x(ux)(1 - u) + 8(kq) g_{\alpha 0} g_{\beta 0} [u(1 - x)](ux)(1 - u) \\
&+ 24(kq) g_{\alpha 0} g_{\beta 0} u(ux)[u(1 - x)] + 4(kq) g_{\alpha 0} g_{\beta 0} (1 - x)(ux)(1 - u) \\
&- 8(kq) g_{\alpha\beta} [u(1 - x)](ux)(1 - u) - 36(kq) g_{\alpha\beta} u(ux)[u(1 - x)] \\
&+ 3(kq) g_{\alpha\beta} u(ux) + 3(kq) g_{\alpha\beta} u[u(1 - x)] + 3(kq) g_{\alpha\beta} u \\
&- 4(kq) g_{\alpha\beta} (1 - x)(ux)(1 - u) + 12k_\alpha k_\beta u(ux)^2 + 6k_\alpha k_\beta u(ux) \\
&- 2q^2 g_{\alpha 0} g_{\beta 0} s[u(1 - x)](1 - u) - 6q^2 g_{\alpha 0} g_{\beta 0} su[u(1 - x)] \\
&- q^2 g_{\alpha 0} g_{\beta 0} s(1 - x)(1 - u) + 2q^2 g_{\alpha 0} g_{\beta 0} [u(1 - x)][u(1 - x)](1 - u) \\
&+ 6q^2 g_{\alpha 0} g_{\beta 0} u[u(1 - x)]^2 + q^2 g_{\alpha 0} g_{\beta 0} (1 - x)[u(1 - x)](1 - u) \\
&+ 2q^2 g_{\alpha\beta} s[u(1 - x)](1 - u) + 6q^2 g_{\alpha\beta} su[u(1 - x)] + q^2 g_{\alpha\beta} s(1 - x)(1 - u) \\
&- 2q^2 g_{\alpha\beta} [u(1 - x)][u(1 - x)](1 - u) - 12q^2 g_{\alpha\beta} u[u(1 - x)]^2 + 3q^2 g_{\alpha\beta} u[u(1 - x)] \\
&- q^2 g_{\alpha\beta} (1 - x)[u(1 - x)](1 - u) + 12q_0^2 g_{\alpha\beta} u[u(1 - x)]^2 + 6q_0^2 g_{\alpha\beta} u[u(1 - x)],
\end{aligned}$$

$$\begin{aligned}
\tilde{L}_{\alpha\beta 00}^{(2)sub} = & -k^4 g_{\alpha\beta} v^2(ux)^2 + k^4 g_{\alpha\beta}(ux)^4 - k^4 g_{\alpha\beta}(ux)^3 \quad (22) \\
& + 4k^2(kq)g_{\alpha\beta}v(ux)^2[u(1-x)] + 4k^2(kq)g_{\alpha\beta}(ux)^3[u(1-x)] - k^2(kq)g_{\alpha\beta}(ux)^3 \\
& - 3k^2(kq)g_{\alpha\beta}(ux)^2[u(1-x)] + k^2(kq)g_{\alpha\beta}(ux)^2 - 2k^2 k_\alpha k_\beta(ux)^4 + 2k^2 k_\alpha k_\beta(ux)^3 \\
& - 2k^2 k_\alpha q_0 g_{\beta 0}(ux)^3[u(1-x)] - 2k^2 k_\beta q_0 g_{\alpha 0}(ux)^3[u(1-x)] \\
& - 2k^2 q^2 g_{\alpha\beta} s v(ux)[u(1-x)] + 2k^2 q^2 g_{\alpha\beta}(ux)^2[u(1-x)]^2 - k^2 q^2 g_{\alpha\beta}(ux)^2[u(1-x)] \\
& - k^2 q^2 g_{\alpha\beta}(ux)[u(1-x)]^2 + 2k^2 q^2 g_{\alpha\beta}(ux)[u(1-x)] - 2k^2 q_0^2 g_{\alpha\beta}(ux)^2[u(1-x)]^2 \\
& - 2k^2 q_0^2 g_{\alpha\beta}(ux)^2[u(1-x)] + 2k^2 g_{\alpha\beta} m^2 v(ux) - 2k^2 g_{\alpha\beta} m^2(ux)^2 + k^2 g_{\alpha\beta} m^2(ux) \\
& - 2(kq)^2 g_{\alpha\beta}(ux)^2[u(1-x)] - 2(kq)^2 g_{\alpha\beta}(ux)[u(1-x)]^2 \\
& - 4(kq)k_\alpha k_\beta(ux)^3[u(1-x)] - 4(kq)k_\alpha q_0 g_{\beta 0}(ux)^2[u(1-x)]^2 \\
& - 4(kq)k_\beta q_0 g_{\alpha 0}(ux)^2[u(1-x)]^2 + 4(kq)q^2 g_{\alpha\beta} s(ux)[u(1-x)]^2 \\
& + 4(kq)q^2 g_{\alpha\beta}(ux)[u(1-x)]^3 - 3(kq)q^2 g_{\alpha\beta}(ux)[u(1-x)]^2 - (kq)q^2 g_{\alpha\beta}[u(1-x)]^3 \\
& + (kq)q^2 g_{\alpha\beta}[u(1-x)]^2 - 4(kq)q_0^2 g_{\alpha\beta}(ux)[u(1-x)]^3 - 8(kq)g_{\alpha\beta} m^2(ux)[u(1-x)] \\
& + (kq)g_{\alpha\beta} m^2(ux) + (kq)g_{\alpha\beta} m^2[u(1-x)] + (kq)g_{\alpha\beta} m^2 - 2k_\alpha k_\beta q^2(ux)^2[u(1-x)]^2 \\
& - 2k_\alpha k_\beta q^2(ux)[u(1-x)]^2 + 8k_\alpha k_\beta q_0^2(ux)^2[u(1-x)]^2 + 2k_\alpha k_\beta m^2(ux)^2 \\
& + 2k_\alpha k_\beta m^2(ux) - 2k_\alpha q^2 q_0 g_{\beta 0}(ux)[u(1-x)]^3 + 2k_\alpha q_0 g_{\beta 0} m^2(ux)[u(1-x)] \\
& - 2k_\beta q^2 q_0 g_{\alpha 0}(ux)[u(1-x)]^3 + 2k_\beta q_0 g_{\alpha 0} m^2(ux)[u(1-x)] - q^4 g_{\alpha\beta} s^2[u(1-x)]^2 \\
& + q^4 g_{\alpha\beta}[u(1-x)]^4 - q^4 g_{\alpha\beta}[u(1-x)]^3 - 2q^2 q_0^2 g_{\alpha\beta}[u(1-x)]^4 \\
& + 2q^2 q_0^2 g_{\alpha\beta}[u(1-x)]^3 + 2q^2 g_{\alpha\beta} m^2 s[u(1-x)] - 2q^2 g_{\alpha\beta} m^2[u(1-x)]^2 \\
& + q^2 g_{\alpha\beta} m^2[u(1-x)] + 2q_0^2 g_{\alpha\beta} m^2[u(1-x)]^2 + 2q_0^2 g_{\alpha\beta} m^2[u(1-x)], \\
v = & 1 - ux, \quad s = 1 - u(1 - x).
\end{aligned}$$

3.2 CROSSED DIAGRAM

The symmetrized expression for the crossed diagram tensor in eq.(4) has the form

$$\begin{aligned}
\tilde{C}_{\alpha\beta 00} = & -\frac{1}{D_1 D_2 D_3 D_4} \{ 2k^2 p^2 g_{\alpha 0} g_{\beta 0} - k^2 p^2 g_{\alpha\beta} - 2k^2 (pq) g_{\alpha 0} g_{\beta 0} \\
& + k^2 (pq) g_{\alpha\beta} - 2k^2 p_0 p_\alpha g_{\beta 0} - 2k^2 p_0 p_\beta g_{\alpha 0} + 2k^2 p_\alpha p_\beta + k^2 p_\alpha q_0 g_{\beta 0} + k^2 p_\beta q_0 g_{\alpha 0} \\
& - 2k^2 g_{\alpha 0} g_{\beta 0} m^2 + k^2 g_{\alpha\beta} m^2 - 2(kp) k_\alpha p_\beta - 2(kp) k_\beta p_\alpha - 4(kp) p^2 g_{\alpha 0} g_{\beta 0} + 2(kp) p^2 g_{\alpha\beta} \\
& + 4(kp)(pq) g_{\alpha 0} g_{\beta 0} - 2(kp)(pq) g_{\alpha\beta} + 4(kp) p_0 p_\alpha g_{\beta 0} + 4(kp) p_0 p_\beta g_{\alpha 0} - 2(kp) p_\alpha q_0 g_{\beta 0} \\
& - 2(kp) p_\beta q_0 g_{\alpha 0} - 2(kp) q^2 g_{\alpha 0} g_{\beta 0} + (kp) q^2 g_{\alpha\beta} - 2(kp) q_0^2 g_{\alpha\beta} + 4(kp) g_{\alpha 0} g_{\beta 0} m^2 \\
& - 2(kp) g_{\alpha\beta} m^2 + (kq) k_\alpha p_\beta + (kq) k_\beta p_\alpha + 2(kq) p^2 g_{\alpha 0} g_{\beta 0} - (kq) p^2 g_{\alpha\beta} - 2(kq) p_0 p_\alpha g_{\beta 0} \\
& - 2(kq) p_0 p_\beta g_{\alpha 0} + 2(kq) p_0 q_0 g_{\alpha\beta} - 2(kq) g_{\alpha 0} g_{\beta 0} m^2 + (kq) g_{\alpha\beta} m^2 + 2k_\alpha k_\beta p^2 \\
& - 2k_\alpha k_\beta (pq) - 2k_\alpha k_\beta m^2 + 2k_\alpha p^2 p_0 g_{\beta 0} - k_\alpha p^2 q_0 g_{\beta 0} - 2k_\alpha (pq) p_0 g_{\beta 0} - 4k_\alpha p_0^2 p_\beta \\
& + 4k_\alpha p_0 p_\beta q_0 + k_\alpha p_0 q^2 g_{\beta 0} - 2k_\alpha p_0 g_{\beta 0} m^2 + k_\alpha q_0 g_{\beta 0} m^2 + 2k_\beta p^2 p_0 g_{\alpha 0} - k_\beta p^2 q_0 g_{\alpha 0} \\
& - 2k_\beta (pq) p_0 g_{\alpha 0} - 4k_\beta p_0^2 p_\alpha + 4k_\beta p_0 p_\alpha q_0 + k_\beta p_0 q^2 g_{\alpha 0} - 2k_\beta p_0 g_{\alpha 0} m^2 + k_\beta q_0 g_{\alpha 0} m^2 \\
& + 2p^4 g_{\alpha 0} g_{\beta 0} - p^4 g_{\alpha\beta} - 4p^2 (pq) g_{\alpha 0} g_{\beta 0} + 2p^2 (pq) g_{\alpha\beta} - 4p^2 p_0 p_\alpha g_{\beta 0} - 4p^2 p_0 p_\beta g_{\alpha 0} \\
& + 2p^2 p_\alpha q_0 g_{\beta 0} + 2p^2 p_\beta q_0 g_{\alpha 0} + 2p^2 q^2 g_{\alpha 0} g_{\beta 0} - p^2 q^2 g_{\alpha\beta} + 2p^2 q_0^2 g_{\alpha\beta} - 4p^2 g_{\alpha 0} g_{\beta 0} m^2 \\
& + 2p^2 g_{\alpha\beta} m^2 + 4(pq) p_0 p_\alpha g_{\beta 0} + 4(pq) p_0 p_\beta g_{\alpha 0} - 4(pq) p_0 q_0 g_{\alpha\beta} + 4(pq) g_{\alpha 0} g_{\beta 0} m^2 \\
& - 2(pq) g_{\alpha\beta} m^2 + 8p_0^2 p_\alpha p_\beta + 2p_0^2 q^2 g_{\alpha\beta} - 8p_0 p_\alpha p_\beta q_0 - 2p_0 p_\alpha q^2 g_{\beta 0} + 4p_0 p_\alpha g_{\beta 0} m^2 \\
& - 2p_0 p_\beta q^2 g_{\alpha 0} + 4p_0 p_\beta g_{\alpha 0} m^2 - 2p_\alpha q_0 g_{\beta 0} m^2 - 2p_\beta q_0 g_{\alpha 0} m^2 - 2q^2 g_{\alpha 0} g_{\beta 0} m^2 \\
& + q^2 g_{\alpha\beta} m^2 - 2q_0^2 g_{\alpha\beta} m^2 + 2g_{\alpha 0} g_{\beta 0} m^4 - g_{\alpha\beta} m^4 \}.
\end{aligned}
\tag{23}$$

Next we combine denominators and perform momentum integration. Unlike the case of the ladder diagram there are four different denominators now and an extra Feynman parameter z is needed to combine denominators

$$\begin{aligned}
(1-z)D_4 + z[(1-u)D_1 + u[xD_2 + (1-x)D_3]] &= p''^2 - \Delta_z, \\
p'' &= p - k[1 - z(1 - ux)] - q[1 - z(1 - u(1 - x))],
\end{aligned}
\tag{24}$$

$$\Delta_z = z\Delta - z(1-z)[k(1-ux) + q(1-u(1-x))]^2 + (1-z)m^2,$$

$$\frac{1}{D_1 D_2 D_3 D_4} = 6 \int_0^1 dx \int_0^1 du \int_0^1 dz u z^2 \frac{1}{(p'^2 - \Delta_z)^4}.$$

Note that $\Delta_{|z=1} = \Delta$ and $p'_{|z=1} = p'$.

After the shift in the crossed diagram we have

$$\begin{aligned} \tilde{C}_{\alpha\beta 00} = & -6 \int_0^1 dx \int_0^1 du \int_0^1 dz u z^2 \int \frac{d^4 p''}{\pi^2 i} \left\{ \frac{2}{3} p''^4 \tilde{C}_{\alpha\beta 00}^{(0)} \right. \\ & \left. + \frac{1}{2} p''^2 \tilde{C}_{\alpha\beta 00}^{(1)} + \tilde{C}_{\alpha\beta 00}^{(2)} \right\} \frac{1}{(p'^2 - \Delta_z)^4}, \end{aligned} \quad (25)$$

where

$$\tilde{C}_{\alpha\beta 00}^{(0)} = -g_{\alpha 0} g_{\beta 0} + g_{\alpha\beta} \equiv -\tilde{L}_{\alpha\beta 00}^{(0)}, \quad (26)$$

$$\tilde{C}_{\alpha\beta 00}^{(1)} = 8k^2 g_{\alpha 0} g_{\beta 0} a^2 - 8k^2 g_{\alpha 0} g_{\beta 0} a + 2k^2 g_{\alpha 0} g_{\beta 0} - 6k^2 g_{\alpha\beta} a^2 + 6k^2 g_{\alpha\beta} a \quad (27)$$

$$\begin{aligned} & -k^2 g_{\alpha\beta} + 16(kq) g_{\alpha 0} g_{\beta 0} ab - 8(kq) g_{\alpha 0} g_{\beta 0} a - 8(kq) g_{\alpha 0} g_{\beta 0} b + 4(kq) g_{\alpha 0} g_{\beta 0} \\ & -12(kq) g_{\alpha\beta} ab + 6(kq) g_{\alpha\beta} a + 6(kq) g_{\alpha\beta} b - 3(kq) g_{\alpha\beta} + 4k_\alpha k_\beta a^2 - 4k_\alpha k_\beta a \\ & + 2k_\alpha k_\beta - 8k_\alpha q_0 g_{\beta 0} ab + 4k_\alpha q_0 g_{\beta 0} a + 4k_\alpha q_0 g_{\beta 0} b - 2k_\alpha q_0 g_{\beta 0} - 8k_\beta q_0 g_{\alpha 0} ab \\ & + 4k_\beta q_0 g_{\alpha 0} a + 4k_\beta q_0 g_{\alpha 0} b - 2k_\beta q_0 g_{\alpha 0} + 8q^2 g_{\alpha 0} g_{\beta 0} b^2 - 8q^2 g_{\alpha 0} g_{\beta 0} b + 2q^2 g_{\alpha 0} g_{\beta 0} \\ & - 6q^2 g_{\alpha\beta} b^2 + 6q^2 g_{\alpha\beta} b - q^2 g_{\alpha\beta} + 4q_0^2 g_{\alpha\beta} b^2 - 4q_0^2 g_{\alpha\beta} b + 2q_0^2 g_{\alpha\beta} - 4g_{\alpha 0} g_{\beta 0} m^2 \\ & + 4g_{\alpha\beta} m^2, \end{aligned}$$

$$\tilde{C}_{\alpha\beta 00}^{(2)} = 2k^4 g_{\alpha 0} g_{\beta 0} a^4 - 4k^4 g_{\alpha 0} g_{\beta 0} a^3 + 2k^4 g_{\alpha 0} g_{\beta 0} a^2 - k^4 g_{\alpha\beta} a^4 + 2k^4 g_{\alpha\beta} a^3 \quad (28)$$

$$\begin{aligned} & -k^4 g_{\alpha\beta} a^2 + 8k^2(kq) g_{\alpha 0} g_{\beta 0} a^3 b - 4k^2(kq) g_{\alpha 0} g_{\beta 0} a^3 - 12k^2(kq) g_{\alpha 0} g_{\beta 0} a^2 b \\ & + 6k^2(kq) g_{\alpha 0} g_{\beta 0} a^2 + 4k^2(kq) g_{\alpha 0} g_{\beta 0} ab - 2k^2(kq) g_{\alpha 0} g_{\beta 0} a - 4k^2(kq) g_{\alpha\beta} a^3 b \\ & + 2k^2(kq) g_{\alpha\beta} a^3 + 6k^2(kq) g_{\alpha\beta} a^2 b - 3k^2(kq) g_{\alpha\beta} a^2 - 2k^2(kq) g_{\alpha\beta} ab + k^2(kq) g_{\alpha\beta} a \\ & - 4k^2 k_\alpha q_0 g_{\beta 0} a^3 b + 2k^2 k_\alpha q_0 g_{\beta 0} a^3 + 6k^2 k_\alpha q_0 g_{\beta 0} a^2 b - 3k^2 k_\alpha q_0 g_{\beta 0} a^2 - 2k^2 k_\alpha q_0 g_{\beta 0} ab \end{aligned}$$

$$\begin{aligned}
& +k^2k_\alpha q_0g_{\beta 0}a-4k^2k_\beta q_0g_{\alpha 0}a^3b+2k^2k_\beta q_0g_{\alpha 0}a^3+6k^2k_\beta q_0g_{\alpha 0}a^2b-3k^2k_\beta q_0g_{\alpha 0}a^2 \\
& \quad -2k^2k_\beta q_0g_{\alpha 0}ab+k^2k_\beta q_0g_{\alpha 0}a+4k^2q^2g_{\alpha 0}g_{\beta 0}a^2b^2-4k^2q^2g_{\alpha 0}g_{\beta 0}a^2b \\
& \quad +2k^2q^2g_{\alpha 0}g_{\beta 0}a^2-4k^2q^2g_{\alpha 0}g_{\beta 0}ab^2+4k^2q^2g_{\alpha 0}g_{\beta 0}ab-2k^2q^2g_{\alpha 0}g_{\beta 0}a \\
& \quad +2k^2q^2g_{\alpha 0}g_{\beta 0}b^2-2k^2q^2g_{\alpha 0}g_{\beta 0}b-2k^2q^2g_{\alpha\beta}a^2b^2+2k^2q^2g_{\alpha\beta}a^2b-k^2q^2g_{\alpha\beta}a^2 \\
& \quad +2k^2q^2g_{\alpha\beta}ab^2-2k^2q^2g_{\alpha\beta}ab+k^2q^2g_{\alpha\beta}a-k^2q^2g_{\alpha\beta}b^2+k^2q^2g_{\alpha\beta}b+2k^2q_0^2g_{\alpha\beta}a^2 \\
& \quad -2k^2q_0^2g_{\alpha\beta}a-4k^2g_{\alpha 0}g_{\beta 0}a^2m^2+4k^2g_{\alpha 0}g_{\beta 0}am^2-2k^2g_{\alpha 0}g_{\beta 0}m^2+2k^2g_{\alpha\beta}a^2m^2 \\
& \quad -2k^2g_{\alpha\beta}am^2+k^2g_{\alpha\beta}m^2+8(kq)^2g_{\alpha 0}g_{\beta 0}a^2b^2-8(kq)^2g_{\alpha 0}g_{\beta 0}a^2b-8(kq)^2g_{\alpha 0}g_{\beta 0}ab^2 \\
& \quad +8(kq)^2g_{\alpha 0}g_{\beta 0}ab-4(kq)^2g_{\alpha\beta}a^2b^2+4(kq)^2g_{\alpha\beta}a^2b+4(kq)^2g_{\alpha\beta}ab^2-4(kq)^2g_{\alpha\beta}ab \\
& \quad -8(kq)k_\alpha q_0g_{\beta 0}a^2b^2+8(kq)k_\alpha q_0g_{\beta 0}a^2b+8(kq)k_\alpha q_0g_{\beta 0}ab^2-8(kq)k_\alpha q_0g_{\beta 0}ab \\
& \quad -8(kq)k_\beta q_0g_{\alpha 0}a^2b^2+8(kq)k_\beta q_0g_{\alpha 0}a^2b+8(kq)k_\beta q_0g_{\alpha 0}ab^2-8(kq)k_\beta q_0g_{\alpha 0}ab \\
& \quad +8(kq)q^2g_{\alpha 0}g_{\beta 0}ab^3-12(kq)q^2g_{\alpha 0}g_{\beta 0}ab^2+4(kq)q^2g_{\alpha 0}g_{\beta 0}ab-4(kq)q^2g_{\alpha 0}g_{\beta 0}b^3 \\
& \quad +6(kq)q^2g_{\alpha 0}g_{\beta 0}b^2-2(kq)q^2g_{\alpha 0}g_{\beta 0}b-4(kq)q^2g_{\alpha\beta}ab^3+6(kq)q^2g_{\alpha\beta}ab^2 \\
& \quad -2(kq)q^2g_{\alpha\beta}ab+2(kq)q^2g_{\alpha\beta}b^3-3(kq)q^2g_{\alpha\beta}b^2+(kq)q^2g_{\alpha\beta}b-8(kq)g_{\alpha 0}g_{\beta 0}abm^2 \\
& \quad +4(kq)g_{\alpha 0}g_{\beta 0}am^2+4(kq)g_{\alpha 0}g_{\beta 0}bm^2-2(kq)g_{\alpha 0}g_{\beta 0}m^2+4(kq)g_{\alpha\beta}abm^2 \\
& \quad -2(kq)g_{\alpha\beta}am^2-2(kq)g_{\alpha\beta}bm^2+(kq)g_{\alpha\beta}m^2+2k_\alpha k_\beta q^2b^2-2k_\alpha k_\beta q^2b \\
& \quad +8k_\alpha k_\beta q_0^2a^2b^2-8k_\alpha k_\beta q_0^2a^2b-8k_\alpha k_\beta q_0^2ab^2+8k_\alpha k_\beta q_0^2ab-2k_\alpha k_\beta m^2 \\
& \quad -4k_\alpha q^2q_0g_{\beta 0}ab^3+6k_\alpha q^2q_0g_{\beta 0}ab^2-2k_\alpha q^2q_0g_{\beta 0}ab+2k_\alpha q^2q_0g_{\beta 0}b^3-3k_\alpha q^2q_0g_{\beta 0}b^2 \\
& \quad +k_\alpha q^2q_0g_{\beta 0}b+4k_\alpha q_0g_{\beta 0}abm^2-2k_\alpha q_0g_{\beta 0}am^2-2k_\alpha q_0g_{\beta 0}bm^2+k_\alpha q_0g_{\beta 0}m^2 \\
& \quad -4k_\beta q^2q_0g_{\alpha 0}ab^3+6k_\beta q^2q_0g_{\alpha 0}ab^2-2k_\beta q^2q_0g_{\alpha 0}ab+2k_\beta q^2q_0g_{\alpha 0}b^3-3k_\beta q^2q_0g_{\alpha 0}b^2 \\
& \quad +k_\beta q^2q_0g_{\alpha 0}b+4k_\beta q_0g_{\alpha 0}abm^2-2k_\beta q_0g_{\alpha 0}am^2-2k_\beta q_0g_{\alpha 0}bm^2+k_\beta q_0g_{\alpha 0}m^2 \\
& \quad +2q^4g_{\alpha 0}g_{\beta 0}b^4-4q^4g_{\alpha 0}g_{\beta 0}b^3+2q^4g_{\alpha 0}g_{\beta 0}b^2-q^4g_{\alpha\beta}b^4+2q^4g_{\alpha\beta}b^3-q^4g_{\alpha\beta}b^2 \\
& \quad -4q^2g_{\alpha 0}g_{\beta 0}b^2m^2+4q^2g_{\alpha 0}g_{\beta 0}bm^2-2q^2g_{\alpha 0}g_{\beta 0}m^2+2q^2g_{\alpha\beta}b^2m^2-2q^2g_{\alpha\beta}bm^2 \\
& \quad +q^2g_{\alpha\beta}m^2-2q_0^2g_{\alpha\beta}m^2+2g_{\alpha 0}g_{\beta 0}m^4-g_{\alpha\beta}m^4,
\end{aligned}$$

$$a = 1 - z(1 - ux),$$

$$b = 1 - z[1 - u(1 - x)].$$

Momentum integration leads to

$$\begin{aligned} \tilde{C}_{\alpha\beta 00} = & \quad (29) \\ -6 \int_0^1 dx \int_0^1 du \int_0^1 dz u z^2 & \left\{ \frac{2}{3} \left(\log \frac{\Lambda^2}{\Delta} - \frac{11}{6} \right) \tilde{C}_{\alpha\beta 00}^{(0)} - \frac{1}{6\Delta_z} \tilde{C}_{\alpha\beta 00}^{(1)} + \frac{1}{6\Delta_z^2} \tilde{C}_{\alpha\beta 00}^{(2)} \right\}. \end{aligned}$$

Again we use integration by parts over z to separate the ultraviolet divergent term

$$\begin{aligned} \int_0^1 dz z^2 \log \frac{\Lambda^2}{\Delta_z} &= -\frac{1-z^3}{3} \log \frac{\Lambda^2}{\Delta_z} \Big|_0^1 \\ &- \int_0^1 \frac{dz}{3\Delta_z} (1-z^3) \{ (\Delta - m^2) - (1-2z)[k(1-ux) + q(1-u(1-x))]^2 \} \\ &= \frac{1}{3} \log \frac{\Lambda^2}{m^2} - \int_0^1 \frac{dz}{3\Delta_z} (1-z^3) \{ (\Delta - m^2) - (1-2z)[k(1-ux) + q(1-u(1-x))]^2 \}. \end{aligned} \quad (30)$$

Subtraction in the logarithmic term at zero momenta is performed by omitting the log as well as the constant term after integration by parts, as was done for the ladder diagram. Again this subtraction does not exhaust all necessary subtractions. The function $\tilde{C}_{\alpha\beta 00}^{(1)}$ contains terms $(-4g_{\alpha 0}g_{\beta 0}m^2 + 4g_{\alpha\beta}m^2)$ and function $\tilde{C}_{\alpha\beta 00}^{(2)}$ contains terms $2g_{\alpha 0}g_{\beta 0}m^4 - g_{\alpha\beta}m^4$ which also require subtractions.

To perform subtractions this time it is sufficient to replace

$$\begin{aligned} \frac{1}{\Delta_z} &\rightarrow \frac{1}{\Delta_z(k, q)} - \frac{1}{\Delta_z(0, 0)} \\ &= \frac{z(m^2 - \Delta) + z(1-z)[k(1-ux) + q(1-u(1-x))]^2}{m^2\Delta_z}, \\ \frac{1}{\Delta_z^2} &\rightarrow \frac{1}{\Delta_z^2(k, q)} - \frac{1}{\Delta_z^2(0, 0)} = \left(\frac{1}{\Delta_z(k, q)} - \frac{1}{\Delta_z(0, 0)} \right) \left(\frac{1}{\Delta_z(k, q)} + \frac{1}{\Delta_z(0, 0)} \right) \\ &= (m^2 + \Delta_z) \frac{z(m^2 - \Delta) + z(1-z)[k(1-ux) + q(1-u(1-x))]^2}{m^4\Delta_z^2}. \end{aligned} \quad (31)$$

in the denominators of the terms just discussed.

After subtraction we obtain

$$\tilde{C}_{\alpha\beta 00}^{sub} = \quad (32)$$

$$- \int_0^1 dx \int_0^1 du \int_0^1 dz \left\{ \frac{u}{3\Delta_z} \tilde{C}_{\alpha\beta 00}^{(1)sub} + \frac{uz^2}{\Delta_z^2} \tilde{C}_{\alpha\beta 00}^{(2)sub} \right\},$$

where

$$\begin{aligned} \tilde{C}_{\alpha\beta 00}^{(1)sub} = & -24k^2 g_{\alpha 0} g_{\beta 0} a^2 z^2 + 24k^2 g_{\alpha 0} g_{\beta 0} a z^2 \quad (33) \\ & + 12k^2 g_{\alpha 0} g_{\beta 0} v^2 z^3 (1-z) + 8k^2 g_{\alpha 0} g_{\beta 0} v z^3 (ux) + 4k^2 g_{\alpha 0} g_{\beta 0} v (ux) \\ & + 4k^2 g_{\alpha 0} g_{\beta 0} z^4 (ux)^2 - 8k^2 g_{\alpha 0} g_{\beta 0} z^4 (ux) + 4k^2 g_{\alpha 0} g_{\beta 0} z^4 - 4k^2 g_{\alpha 0} g_{\beta 0} z^3 (ux)^2 (1-z) \\ & + 8k^2 g_{\alpha 0} g_{\beta 0} z^3 (ux) (1-z) - 4k^2 g_{\alpha 0} g_{\beta 0} z^3 (1-z) - 6k^2 g_{\alpha 0} g_{\beta 0} z^2 - 4k^2 g_{\alpha 0} g_{\beta 0} z (ux)^2 \\ & + 8k^2 g_{\alpha 0} g_{\beta 0} z (ux) - 4k^2 g_{\alpha 0} g_{\beta 0} z + 4k^2 g_{\alpha 0} g_{\beta 0} (ux)^2 (1-z) - 8k^2 g_{\alpha 0} g_{\beta 0} (ux) (1-z) \\ & + 4k^2 g_{\alpha 0} g_{\beta 0} (1-z) + 18k^2 g_{\alpha\beta} a^2 z^2 - 18k^2 g_{\alpha\beta} a z^2 - 12k^2 g_{\alpha\beta} v^2 z^3 (1-z) \\ & - 8k^2 g_{\alpha\beta} v z^3 (ux) - 4k^2 g_{\alpha\beta} v (ux) - 4k^2 g_{\alpha\beta} z^4 (ux)^2 + 8k^2 g_{\alpha\beta} z^4 (ux) - 4k^2 g_{\alpha\beta} z^4 \\ & + 4k^2 g_{\alpha\beta} z^3 (ux)^2 (1-z) - 8k^2 g_{\alpha\beta} z^3 (ux) (1-z) + 4k^2 g_{\alpha\beta} z^3 (1-z) + 3k^2 g_{\alpha\beta} z^2 \\ & + 4k^2 g_{\alpha\beta} z (ux)^2 - 8k^2 g_{\alpha\beta} z (ux) + 4k^2 g_{\alpha\beta} z - 4k^2 g_{\alpha\beta} (ux)^2 (1-z) \\ & + 8k^2 g_{\alpha\beta} (ux) (1-z) - 4k^2 g_{\alpha\beta} (1-z) - 48(kq) g_{\alpha 0} g_{\beta 0} a b z^2 + 24(kq) g_{\alpha 0} g_{\beta 0} a z^2 \\ & + 24(kq) g_{\alpha 0} g_{\beta 0} b z^2 + 24(kq) g_{\alpha 0} g_{\beta 0} s v z^3 (1-z) + 8(kq) g_{\alpha 0} g_{\beta 0} z^4 (ux) [u(1-x)] \\ & - 8(kq) g_{\alpha 0} g_{\beta 0} z^4 (ux) - 8(kq) g_{\alpha 0} g_{\beta 0} z^4 [u(1-x)] + 8(kq) g_{\alpha 0} g_{\beta 0} z^4 \\ & - 8(kq) g_{\alpha 0} g_{\beta 0} z^3 (ux) [u(1-x)] (1-z) - 16(kq) g_{\alpha 0} g_{\beta 0} z^3 (ux) [u(1-x)] \\ & + 8(kq) g_{\alpha 0} g_{\beta 0} z^3 (ux) (1-z) + 8(kq) g_{\alpha 0} g_{\beta 0} z^3 [u(1-x)] (1-z) \\ & - 8(kq) g_{\alpha 0} g_{\beta 0} z^3 (1-z) - 12(kq) g_{\alpha 0} g_{\beta 0} z^2 - 8(kq) g_{\alpha 0} g_{\beta 0} z (ux) [u(1-x)] \\ & + 8(kq) g_{\alpha 0} g_{\beta 0} z (ux) + 8(kq) g_{\alpha 0} g_{\beta 0} z [u(1-x)] - 8(kq) g_{\alpha 0} g_{\beta 0} z \\ & + 8(kq) g_{\alpha 0} g_{\beta 0} (ux) [u(1-x)] (1-z) - 8(kq) g_{\alpha 0} g_{\beta 0} (ux) [u(1-x)] \\ & - 8(kq) g_{\alpha 0} g_{\beta 0} (ux) (1-z) - 8(kq) g_{\alpha 0} g_{\beta 0} [u(1-x)] (1-z) + 8(kq) g_{\alpha 0} g_{\beta 0} (1-z) \\ & + 36(kq) g_{\alpha\beta} a b z^2 - 18(kq) g_{\alpha\beta} a z^2 - 18(kq) g_{\alpha\beta} b z^2 - 24(kq) g_{\alpha\beta} s v z^3 (1-z) \\ & - 8(kq) g_{\alpha\beta} z^4 (ux) [u(1-x)] + 8(kq) g_{\alpha\beta} z^4 (ux) + 8(kq) g_{\alpha\beta} z^4 [u(1-x)] \end{aligned}$$

$$\begin{aligned}
& -8(kq)g_{\alpha\beta}z^4 + 8(kq)g_{\alpha\beta}z^3(ux)[u(1-x)](1-z) + 16(kq)g_{\alpha\beta}z^3(ux)[u(1-x)] \\
& -8(kq)g_{\alpha\beta}z^3(ux)(1-z) - 8(kq)g_{\alpha\beta}z^3[u(1-x)](1-z) + 8(kq)g_{\alpha\beta}z^3(1-z) \\
& +9(kq)g_{\alpha\beta}z^2 + 8(kq)g_{\alpha\beta}z(ux)[u(1-x)] - 8(kq)g_{\alpha\beta}z(ux) - 8(kq)g_{\alpha\beta}z[u(1-x)] \\
& +8(kq)g_{\alpha\beta}z - 8(kq)g_{\alpha\beta}(ux)[u(1-x)](1-z) + 8(kq)g_{\alpha\beta}(ux)[u(1-x)] \\
& +8(kq)g_{\alpha\beta}(ux)(1-z) + 8(kq)g_{\alpha\beta}[u(1-x)](1-z) - 8(kq)g_{\alpha\beta}(1-z) - 12k_\alpha k_\beta a^2 z^2 \\
& +12k_\alpha k_\beta a z^2 - 6k_\alpha k_\beta z^2 + 24k_\alpha q_0 g_{\beta 0} a b z^2 - 12k_\alpha q_0 g_{\beta 0} a z^2 - 12k_\alpha q_0 g_{\beta 0} b z^2 \\
& +6k_\alpha q_0 g_{\beta 0} z^2 + 24k_\beta q_0 g_{\alpha 0} a b z^2 - 12k_\beta q_0 g_{\alpha 0} a z^2 - 12k_\beta q_0 g_{\alpha 0} b z^2 + 6k_\beta q_0 g_{\alpha 0} z^2 \\
& -24q^2 g_{\alpha 0} g_{\beta 0} b^2 z^2 + 24q^2 g_{\alpha 0} g_{\beta 0} b z^2 + 12q^2 g_{\alpha 0} g_{\beta 0} s^2 z^3 (1-z) \\
& +8q^2 g_{\alpha 0} g_{\beta 0} s z^3 [u(1-x)] + 4q^2 g_{\alpha 0} g_{\beta 0} s [u(1-x)] + 4q^2 g_{\alpha 0} g_{\beta 0} z^4 [u(1-x)]^2 \\
& -8q^2 g_{\alpha 0} g_{\beta 0} z^4 [u(1-x)] + 4q^2 g_{\alpha 0} g_{\beta 0} z^4 - 4q^2 g_{\alpha 0} g_{\beta 0} z^3 [u(1-x)]^2 (1-z) \\
& +8q^2 g_{\alpha 0} g_{\beta 0} z^3 [u(1-x)](1-z) - 4q^2 g_{\alpha 0} g_{\beta 0} z^3 (1-z) - 6q^2 g_{\alpha 0} g_{\beta 0} z^2 \\
& -4q^2 g_{\alpha 0} g_{\beta 0} z [u(1-x)]^2 + 8q^2 g_{\alpha 0} g_{\beta 0} z [u(1-x)] - 4q^2 g_{\alpha 0} g_{\beta 0} z \\
& +4q^2 g_{\alpha 0} g_{\beta 0} [u(1-x)]^2 (1-z) - 8q^2 g_{\alpha 0} g_{\beta 0} [u(1-x)](1-z) + 4q^2 g_{\alpha 0} g_{\beta 0} (1-z) \\
& +18q^2 g_{\alpha\beta} b^2 z^2 - 18q^2 g_{\alpha\beta} b z^2 - 12q^2 g_{\alpha\beta} s^2 z^3 (1-z) - 8q^2 g_{\alpha\beta} s z^3 [u(1-x)] \\
& -4q^2 g_{\alpha\beta} s [u(1-x)] - 4q^2 g_{\alpha\beta} z^4 [u(1-x)]^2 + 8q^2 g_{\alpha\beta} z^4 [u(1-x)] - 4q^2 g_{\alpha\beta} z^4 \\
& +4q^2 g_{\alpha\beta} z^3 [u(1-x)]^2 (1-z) - 8q^2 g_{\alpha\beta} z^3 [u(1-x)](1-z) + 4q^2 g_{\alpha\beta} z^3 (1-z) \\
& +3q^2 g_{\alpha\beta} z^2 + 4q^2 g_{\alpha\beta} z [u(1-x)]^2 - 8q^2 g_{\alpha\beta} z [u(1-x)] + 4q^2 g_{\alpha\beta} z \\
& -4q^2 g_{\alpha\beta} [u(1-x)]^2 (1-z) + 8q^2 g_{\alpha\beta} [u(1-x)](1-z) - 4q^2 g_{\alpha\beta} (1-z) - 12q_0^2 g_{\alpha\beta} b^2 z^2 \\
& +12q_0^2 g_{\alpha\beta} b z^2 - 6q_0^2 g_{\alpha\beta} z^2,
\end{aligned}$$

$$\begin{aligned}
\tilde{C}_{\alpha\beta 00}^{(2)sub} &= 2k^4 g_{\alpha 0} g_{\beta 0} a^4 - 4k^4 g_{\alpha 0} g_{\beta 0} a^3 + 2k^4 g_{\alpha 0} g_{\beta 0} a^2 - 2k^4 g_{\alpha 0} g_{\beta 0} v^4 z^2 (1-z)^2 \\
& -4k^4 g_{\alpha 0} g_{\beta 0} v^3 z^2 (ux)(1-z) - 2k^4 g_{\alpha 0} g_{\beta 0} v^2 z^2 (ux)^2 - k^4 g_{\alpha\beta} a^4 + 2k^4 g_{\alpha\beta} a^3 - k^4 g_{\alpha\beta} a^2 \\
& +k^4 g_{\alpha\beta} v^4 z^2 (1-z)^2 + 2k^4 g_{\alpha\beta} v^3 z^2 (ux)(1-z) + k^4 g_{\alpha\beta} v^2 z^2 (ux)^2
\end{aligned} \tag{34}$$

$$\begin{aligned}
& +8k^2(kq)g_{\alpha 0}g_{\beta 0}a^3b - 4k^2(kq)g_{\alpha 0}g_{\beta 0}a^3 - 12k^2(kq)g_{\alpha 0}g_{\beta 0}a^2b + 6k^2(kq)g_{\alpha 0}g_{\beta 0}a^2 \\
& \quad +4k^2(kq)g_{\alpha 0}g_{\beta 0}ab - 2k^2(kq)g_{\alpha 0}g_{\beta 0}a - 8k^2(kq)g_{\alpha 0}g_{\beta 0}sv^3z^2(1-z)^2 \\
& \quad -8k^2(kq)g_{\alpha 0}g_{\beta 0}sv^2z^2(ux)(1-z) + 8k^2(kq)g_{\alpha 0}g_{\beta 0}v^2z^2(ux)[u(1-x)](1-z) \\
& \quad +8k^2(kq)g_{\alpha 0}g_{\beta 0}vz^2(ux)^2[u(1-x)] - 4k^2(kq)g_{\alpha\beta}a^3b + 2k^2(kq)g_{\alpha\beta}a^3 \\
& \quad +6k^2(kq)g_{\alpha\beta}a^2b - 3k^2(kq)g_{\alpha\beta}a^2 - 2k^2(kq)g_{\alpha\beta}ab + k^2(kq)g_{\alpha\beta}a \\
& \quad +4k^2(kq)g_{\alpha\beta}sv^3z^2(1-z)^2 + 4k^2(kq)g_{\alpha\beta}sv^2z^2(ux)(1-z) \\
& \quad -4k^2(kq)g_{\alpha\beta}v^2z^2(ux)[u(1-x)](1-z) - 4k^2(kq)g_{\alpha\beta}vz^2(ux)^2[u(1-x)] \\
& -4k^2k_{\alpha}q_0g_{\beta 0}a^3b + 2k^2k_{\alpha}q_0g_{\beta 0}a^3 + 6k^2k_{\alpha}q_0g_{\beta 0}a^2b - 3k^2k_{\alpha}q_0g_{\beta 0}a^2 - 2k^2k_{\alpha}q_0g_{\beta 0}ab \\
& +k^2k_{\alpha}q_0g_{\beta 0}a - 4k^2k_{\beta}q_0g_{\alpha 0}a^3b + 2k^2k_{\beta}q_0g_{\alpha 0}a^3 + 6k^2k_{\beta}q_0g_{\alpha 0}a^2b - 3k^2k_{\beta}q_0g_{\alpha 0}a^2 \\
& \quad -2k^2k_{\beta}q_0g_{\alpha 0}ab + k^2k_{\beta}q_0g_{\alpha 0}a + 4k^2q^2g_{\alpha 0}g_{\beta 0}a^2b^2 - 4k^2q^2g_{\alpha 0}g_{\beta 0}a^2b \\
& \quad +2k^2q^2g_{\alpha 0}g_{\beta 0}a^2 - 4k^2q^2g_{\alpha 0}g_{\beta 0}ab^2 + 4k^2q^2g_{\alpha 0}g_{\beta 0}ab - 2k^2q^2g_{\alpha 0}g_{\beta 0}a \\
& \quad +2k^2q^2g_{\alpha 0}g_{\beta 0}b^2 - 2k^2q^2g_{\alpha 0}g_{\beta 0}b - 4k^2q^2g_{\alpha 0}g_{\beta 0}s^2v^2z^2(1-z)^2 \\
& \quad -4k^2q^2g_{\alpha 0}g_{\beta 0}s^2vz^2(ux)(1-z) - 4k^2q^2g_{\alpha 0}g_{\beta 0}sv^2z^2[u(1-x)](1-z) \\
& \quad -4k^2q^2g_{\alpha 0}g_{\beta 0}svz^2(ux)[u(1-x)] - 2k^2q^2g_{\alpha\beta}a^2b^2 + 2k^2q^2g_{\alpha\beta}a^2b - k^2q^2g_{\alpha\beta}a^2 \\
& \quad +2k^2q^2g_{\alpha\beta}ab^2 - 2k^2q^2g_{\alpha\beta}ab + k^2q^2g_{\alpha\beta}a - k^2q^2g_{\alpha\beta}b^2 + k^2q^2g_{\alpha\beta}b \\
& \quad +2k^2q^2g_{\alpha\beta}s^2v^2z^2(1-z)^2 + 2k^2q^2g_{\alpha\beta}s^2vz^2(ux)(1-z) \\
& +2k^2q^2g_{\alpha\beta}sv^2z^2[u(1-x)](1-z) + 2k^2q^2g_{\alpha\beta}svz^2(ux)[u(1-x)] + 2k^2q_0^2g_{\alpha\beta}a^2 \\
& \quad -2k^2q_0^2g_{\alpha\beta}a - 4k^2g_{\alpha 0}g_{\beta 0}a^2m^2 + 4k^2g_{\alpha 0}g_{\beta 0}am^2 + 2k^2g_{\alpha 0}g_{\beta 0}m^2v^2z^2(1-z) \\
& \quad +2k^2g_{\alpha 0}g_{\beta 0}m^2v^2z(1-z)^2 + 2k^2g_{\alpha 0}g_{\beta 0}m^2v^2z(1-z) + 2k^2g_{\alpha 0}g_{\beta 0}m^2vz^2(ux) \\
& +2k^2g_{\alpha 0}g_{\beta 0}m^2vz(ux)(1-z) + 2k^2g_{\alpha 0}g_{\beta 0}m^2vz(ux) - 2k^2g_{\alpha 0}g_{\beta 0}m^2 + 2k^2g_{\alpha\beta}a^2m^2 \\
& -2k^2g_{\alpha\beta}am^2 - k^2g_{\alpha\beta}m^2v^2z^2(1-z) - k^2g_{\alpha\beta}m^2v^2z(1-z)^2 - k^2g_{\alpha\beta}m^2v^2z(1-z) \\
& \quad -k^2g_{\alpha\beta}m^2vz^2(ux) - k^2g_{\alpha\beta}m^2vz(ux)(1-z) - k^2g_{\alpha\beta}m^2vz(ux) + k^2g_{\alpha\beta}m^2 \\
& +8(kq)^2g_{\alpha 0}g_{\beta 0}a^2b^2 - 8(kq)^2g_{\alpha 0}g_{\beta 0}a^2b - 8(kq)^2g_{\alpha 0}g_{\beta 0}ab^2 + 8(kq)^2g_{\alpha 0}g_{\beta 0}ab \\
& \quad -8(kq)^2g_{\alpha 0}g_{\beta 0}s^2v^2z^2(1-z)^2 + 16(kq)^2g_{\alpha 0}g_{\beta 0}svz^2(ux)[u(1-x)](1-z)
\end{aligned}$$

$$\begin{aligned}
& -8(kq)^2 g_{\alpha 0} g_{\beta 0} z^2 (ux)^2 [u(1-x)]^2 - 4(kq)^2 g_{\alpha \beta} a^2 b^2 + 4(kq)^2 g_{\alpha \beta} a^2 b + 4(kq)^2 g_{\alpha \beta} ab^2 \\
& - 4(kq)^2 g_{\alpha \beta} ab + 4(kq)^2 g_{\alpha \beta} s^2 v^2 z^2 (1-z)^2 - 8(kq)^2 g_{\alpha \beta} sv z^2 (ux) [u(1-x)] (1-z) \\
& + 4(kq)^2 g_{\alpha \beta} z^2 (ux)^2 [u(1-x)]^2 - 8(kq) k_{\alpha} q_0 g_{\beta 0} a^2 b^2 + 8(kq) k_{\alpha} q_0 g_{\beta 0} a^2 b \\
& + 8(kq) k_{\alpha} q_0 g_{\beta 0} ab^2 - 8(kq) k_{\alpha} q_0 g_{\beta 0} ab - 8(kq) k_{\beta} q_0 g_{\alpha 0} a^2 b^2 + 8(kq) k_{\beta} q_0 g_{\alpha 0} a^2 b \\
& + 8(kq) k_{\beta} q_0 g_{\alpha 0} ab^2 - 8(kq) k_{\beta} q_0 g_{\alpha 0} ab + 8(kq) q^2 g_{\alpha 0} g_{\beta 0} ab^3 - 12(kq) q^2 g_{\alpha 0} g_{\beta 0} ab^2 \\
& + 4(kq) q^2 g_{\alpha 0} g_{\beta 0} ab - 4(kq) q^2 g_{\alpha 0} g_{\beta 0} b^3 + 6(kq) q^2 g_{\alpha 0} g_{\beta 0} b^2 - 2(kq) q^2 g_{\alpha 0} g_{\beta 0} b \\
& - 8(kq) q^2 g_{\alpha 0} g_{\beta 0} s^3 v z^2 (1-z)^2 - 8(kq) q^2 g_{\alpha 0} g_{\beta 0} s^2 v z^2 [u(1-x)] (1-z) \\
& + 8(kq) q^2 g_{\alpha 0} g_{\beta 0} s^2 z^2 (ux) [u(1-x)] (1-z) + 8(kq) q^2 g_{\alpha 0} g_{\beta 0} s z^2 (ux) [u(1-x)]^2 \\
& - 4(kq) q^2 g_{\alpha \beta} ab^3 + 6(kq) q^2 g_{\alpha \beta} ab^2 - 2(kq) q^2 g_{\alpha \beta} ab + 2(kq) q^2 g_{\alpha \beta} b^3 - 3(kq) q^2 g_{\alpha \beta} b^2 \\
& + (kq) q^2 g_{\alpha \beta} b + 4(kq) q^2 g_{\alpha \beta} s^3 v z^2 (1-z)^2 + 4(kq) q^2 g_{\alpha \beta} s^2 v z^2 [u(1-x)] (1-z) \\
& - 4(kq) q^2 g_{\alpha \beta} s^2 z^2 (ux) [u(1-x)] (1-z) - 4(kq) q^2 g_{\alpha \beta} s z^2 (ux) [u(1-x)]^2 \\
& - 8(kq) g_{\alpha 0} g_{\beta 0} ab m^2 + 4(kq) g_{\alpha 0} g_{\beta 0} am^2 + 4(kq) g_{\alpha 0} g_{\beta 0} bm^2 \\
& + 4(kq) g_{\alpha 0} g_{\beta 0} m^2 sv z^2 (1-z) + 4(kq) g_{\alpha 0} g_{\beta 0} m^2 sv z (1-z)^2 \\
& + 4(kq) g_{\alpha 0} g_{\beta 0} m^2 sv z (1-z) - 4(kq) g_{\alpha 0} g_{\beta 0} m^2 z^2 (ux) [u(1-x)] \\
& - 4(kq) g_{\alpha 0} g_{\beta 0} m^2 z (ux) [u(1-x)] (1-z) - 4(kq) g_{\alpha 0} g_{\beta 0} m^2 z (ux) [u(1-x)]^2 \\
& - 2(kq) g_{\alpha 0} g_{\beta 0} m^2 + 4(kq) g_{\alpha \beta} ab m^2 - 2(kq) g_{\alpha \beta} am^2 - 2(kq) g_{\alpha \beta} bm^2 \\
& - 2(kq) g_{\alpha \beta} m^2 sv z^2 (1-z) - 2(kq) g_{\alpha \beta} m^2 sv z (1-z)^2 - 2(kq) g_{\alpha \beta} m^2 sv z (1-z) \\
& + 2(kq) g_{\alpha \beta} m^2 z^2 (ux) [u(1-x)] + 2(kq) g_{\alpha \beta} m^2 z (ux) [u(1-x)] (1-z) \\
& + 2(kq) g_{\alpha \beta} m^2 z (ux) [u(1-x)] + (kq) g_{\alpha \beta} m^2 + 2k_{\alpha} k_{\beta} q^2 b^2 - 2k_{\alpha} k_{\beta} q^2 b + 8k_{\alpha} k_{\beta} q_0^2 a^2 b^3 \\
& - 8k_{\alpha} k_{\beta} q_0^2 a^2 b - 8k_{\alpha} k_{\beta} q_0^2 ab^2 + 8k_{\alpha} k_{\beta} q_0^2 ab - 2k_{\alpha} k_{\beta} m^2 - 4k_{\alpha} q^2 q_0 g_{\beta 0} ab^3 \\
& + 6k_{\alpha} q^2 q_0 g_{\beta 0} ab^2 - 2k_{\alpha} q^2 q_0 g_{\beta 0} ab + 2k_{\alpha} q^2 q_0 g_{\beta 0} b^3 - 3k_{\alpha} q^2 q_0 g_{\beta 0} b^2 + k_{\alpha} q^2 q_0 g_{\beta 0} b \\
& + 4k_{\alpha} q_0 g_{\beta 0} ab m^2 - 2k_{\alpha} q_0 g_{\beta 0} am^2 - 2k_{\alpha} q_0 g_{\beta 0} bm^2 + k_{\alpha} q_0 g_{\beta 0} m^2 - 4k_{\beta} q^2 q_0 g_{\alpha 0} ab^3 \\
& + 6k_{\beta} q^2 q_0 g_{\alpha 0} ab^2 - 2k_{\beta} q^2 q_0 g_{\alpha 0} ab + 2k_{\beta} q^2 q_0 g_{\alpha 0} b^3 - 3k_{\beta} q^2 q_0 g_{\alpha 0} b^2 + k_{\beta} q^2 q_0 g_{\alpha 0} b \\
& + 4k_{\beta} q_0 g_{\alpha 0} ab m^2 - 2k_{\beta} q_0 g_{\alpha 0} am^2 - 2k_{\beta} q_0 g_{\alpha 0} bm^2 + k_{\beta} q_0 g_{\alpha 0} m^2 + 2q^4 g_{\alpha 0} g_{\beta 0} b^4
\end{aligned}$$

$$\begin{aligned}
& -4q^4 g_{\alpha 0} g_{\beta 0} b^3 + 2q^4 g_{\alpha 0} g_{\beta 0} b^2 - 2q^4 g_{\alpha 0} g_{\beta 0} s^4 z^2 (1-z)^2 \\
& -4q^4 g_{\alpha 0} g_{\beta 0} s^3 z^2 [u(1-x)](1-z) - 2q^4 g_{\alpha 0} g_{\beta 0} s^2 z^2 [u(1-x)]^2 - q^4 g_{\alpha \beta} b^4 + 2q^4 g_{\alpha \beta} b^3 \\
& - q^4 g_{\alpha \beta} b^2 + q^4 g_{\alpha \beta} s^4 z^2 (1-z)^2 + 2q^4 g_{\alpha \beta} s^3 z^2 [u(1-x)](1-z) + q^4 g_{\alpha \beta} s^2 z^2 [u(1-x)]^2 \\
& - 4q^2 g_{\alpha 0} g_{\beta 0} b^2 m^2 + 4q^2 g_{\alpha 0} g_{\beta 0} b m^2 + 2q^2 g_{\alpha 0} g_{\beta 0} m^2 s^2 z^2 (1-z) \\
& + 2q^2 g_{\alpha 0} g_{\beta 0} m^2 s^2 z (1-z)^2 + 2q^2 g_{\alpha 0} g_{\beta 0} m^2 s^2 z (1-z) + 2q^2 g_{\alpha 0} g_{\beta 0} m^2 s^2 z^2 [u(1-x)] \\
& + 2q^2 g_{\alpha 0} g_{\beta 0} m^2 s z [u(1-x)](1-z) + 2q^2 g_{\alpha 0} g_{\beta 0} m^2 s z [u(1-x)] - 2q^2 g_{\alpha 0} g_{\beta 0} m^2 \\
& + 2q^2 g_{\alpha \beta} b^2 m^2 - 2q^2 g_{\alpha \beta} b m^2 - q^2 g_{\alpha \beta} m^2 s^2 z^2 (1-z) - q^2 g_{\alpha \beta} m^2 s^2 z (1-z)^2 \\
& - q^2 g_{\alpha \beta} m^2 s^2 z (1-z) - q^2 g_{\alpha \beta} m^2 s z^2 [u(1-x)] - q^2 g_{\alpha \beta} m^2 s z [u(1-x)](1-z) \\
& - q^2 g_{\alpha \beta} m^2 s z [u(1-x)] + q^2 g_{\alpha \beta} m^2 - 2q_0^2 g_{\alpha \beta} m^2.
\end{aligned}$$

3.3 LOW FREQUENCY LIMIT

We have obtained in eq.(20) and eq.(32) explicit expressions for the contributions to the light by light scattering tensor induced by the ladder and crossed diagrams, respectively. These expressions have the form of the two- and three-dimensional integrals over the Feynman parameters. A very important check of the correctness of all preceding calculations may be performed now. Namely, we are going to compare the low-frequency limit of the expressions above for the light by light scattering tensor with the well known expression given by the Euler-Heisenberg Lagrangian. Leading terms in the expansion of the tensor in eq.(2) over k and q should coincide with the Euler-Heisenberg Lagrangian. Naively, it is far from evident that the expansion of the integrals over Feynman parameters obtained above will start with terms at least quartic in momenta and that these leading terms will be at least quadratic simultaneously in q and k .

Performing the low frequency expansion in eq.(20) and eq.(32) we obtain

$$\begin{aligned}
\mathcal{S}_{\alpha\beta 00}^{low} & \equiv \lim_{k \rightarrow 0, q \rightarrow 0} [2\tilde{\mathcal{L}}_{\alpha\beta 00}^{sub} + \tilde{\mathcal{C}}_{\alpha\beta 00}^{sub}] \quad (35) \\
& = \frac{13}{315m^4} k^2 (kq) g_{\beta 0} g_{\alpha 0} + \frac{38}{315m^4} k^2 (kq) g_{\alpha \beta} + \frac{1}{126m^4} k^2 k_{\beta} q_0 g_{\alpha 0} \\
& + \frac{1}{126m^4} k^2 k_{\alpha} q_0 g_{\beta 0} - \frac{7}{45m^4} k^2 q^2 g_{\beta 0} g_{\alpha 0} + \frac{1}{9m^4} k^2 q^2 g_{\alpha \beta} - \frac{7}{45m^4} k^2 q_0^2 g_{\alpha \beta}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{15m^4}(kq)^2 g_{\beta 0} g_{\alpha 0} - \frac{7}{45m^4}(kq)^2 g_{\alpha \beta} - \frac{4}{35m^4}(kq) k_{\beta} k_{\alpha} - \frac{1}{15m^4}(kq) k_{\beta} q_0 g_{\alpha 0} \\
& - \frac{1}{15m^4}(kq) k_{\alpha} q_0 g_{\beta 0} + \frac{13}{315m^4}(kq) q^2 g_{\beta 0} g_{\alpha 0} + \frac{38}{315m^4}(kq) q^2 g_{\alpha \beta} - \frac{4}{35m^4}(kq) q_0^2 g_{\alpha \beta} \\
& + \frac{1}{5m^4}(kq) g_{\beta 0} g_{\alpha 0} m^2 + \frac{3}{5m^4}(kq) g_{\alpha \beta} m^2 - \frac{7}{45m^4} k_{\beta} k_{\alpha} q^2 + \frac{1}{15m^4} k_{\beta} k_{\alpha} q_0^2 \\
& + \frac{1}{126m^4} k_{\beta} q^2 q_0 g_{\alpha 0} + \frac{1}{30m^4} k_{\beta} q_0 g_{\alpha 0} m^2 + \frac{1}{126m^4} k_{\alpha} q^2 q_0 g_{\beta 0} + \frac{1}{30m^4} k_{\alpha} q_0 g_{\beta 0} m^2.
\end{aligned}$$

Note that some undesirable terms containing too small a number of factors of k or q emerged in the expansion. One should not be too disappointed at this stage since we have performed above a transformation of the light by light scattering tensor taking into account that further integration over k kills all odd factors in k . It is easy to see that all unwanted terms are odd in k and, hence, they disappear after the integration over k .

Really, it is not difficult to calculate second ladder diagram for the light by light scattering separately and to obtain a low frequency asymptote which does not contain unwanted terms at all

$$\begin{aligned}
\tilde{\mathcal{S}}_{\alpha\beta 00}^{low} & \equiv \lim_{k \rightarrow 0, q \rightarrow 0} [\tilde{\mathcal{L}}_{\alpha\beta 00}^{1sub} + \tilde{\mathcal{L}}_{\alpha\beta 00}^{2sub} + \tilde{\mathcal{C}}_{\alpha\beta 00}^{sub}] \quad (36) \\
& = -\frac{7}{45m^4} k^2 q^2 g_{\beta 0} g_{\alpha 0} + \frac{1}{9m^4} k^2 q^2 g_{\alpha \beta} - \frac{7}{45m^4} k^2 q_0^2 g_{\alpha \beta} \\
& + \frac{1}{15m^4} (kq)^2 g_{\beta 0} g_{\alpha 0} - \frac{7}{45m^4} (kq)^2 g_{\alpha \beta} - \frac{1}{15m^4} (kq) k_{\beta} q_0 g_{\alpha 0} \\
& - \frac{1}{15m^4} (kq) k_{\alpha} q_0 g_{\beta 0} - \frac{7}{45m^4} k_{\beta} k_{\alpha} q^2 + \frac{1}{15m^4} k_{\beta} k_{\alpha} q_0^2.
\end{aligned}$$

It is easy to see that this last expression coincides exactly with the one corresponding to the Euler-Heisenberg Lagrangian (for the Euler-Heisenberg expression see, e.g. [4]).

4 CALCULATION OF THE CONTRIBUTION TO THE LAMB SHIFT INDUCED BY THE LIGHT BY LIGHT SCATTERING INSERTION

4.1 LADDER DIAGRAM

We start the calculation of the contribution to the energy shift induced by the ladder diagrams in Fig. 1 by multiplication of the respective part of the light by light scattering tensor by the numerator of the electron factor in eq.(3)³. Thus we obtain

$$L_l = 2A_{\alpha\beta}\tilde{\mathcal{L}}_{\alpha\beta 00}^{sub} \quad (37)$$

$$= -2 \int_0^1 dx \int_0^1 du \left\{ \frac{1-u}{\Delta} F_1^l + \frac{2u(1-u)}{\Delta^2} F_2^l \right\},$$

where

$$F_1^l = -2k^2 q_0 u v x (1-u) - 6k^2 q_0 u v (ux) + 6k^2 q_0 u (ux)^2 - 4k^2 q_0 u (ux) \quad (38)$$

$$- k^2 q_0 v x (1-u) + k^2 q_0 x (1-u) (ux) + 2k^2 q_0 (1-u) (ux)^2 - 4k^2 m u (ux)^2$$

$$+ 2k^2 m u (ux) + 32(kq) q_0 u (ux) [u(1-x)] - 2(kq) q_0 u (ux) - 2(kq) q_0 u [u(1-x)]$$

$$- 2(kq) q_0 u + 4(kq) q_0 x (1-u) (ux) + 8(kq) q_0 (1-u) (ux) [u(1-x)]$$

$$- 8(kq) m u (ux) [u(1-x)] + 2(kq) m u (ux) + 2(kq) m u [u(1-x)] + 2(kq) m u$$

$$- 6q^2 q_0 s u [u(1-x)] - q^2 q_0 s x (1-u) - 2q^2 q_0 s (1-u) [u(1-x)] + 10q^2 q_0 u [u(1-x)]^2$$

$$- 2q^2 q_0 u [u(1-x)] + q^2 q_0 x (1-u) [u(1-x)] + 2q^2 q_0 (1-u) [u(1-x)]^2$$

$$- 4q^2 m u [u(1-x)]^2 + 2q^2 m u [u(1-x)] - 8q_0^3 u [u(1-x)]^2 - 4q_0^3 u [u(1-x)]$$

$$+ 8q_0^2 m u [u(1-x)]^2 + 4q_0^2 m u [u(1-x)],$$

$$F_2^l = k^4 q_0 v^2 (ux)^2 - k^4 m v^2 (ux)^2 + k^4 m (ux)^4 - k^4 m (ux)^3 \quad (39)$$

$$- 4k^2 (kq) q_0 v (ux)^2 [u(1-x)] - 4k^2 (kq) q_0 (ux)^3 [u(1-x)] + k^2 (kq) q_0 (ux)^3$$

$$+ 3k^2 (kq) q_0 (ux)^2 [u(1-x)] - k^2 (kq) q_0 (ux)^2 + 4k^2 (kq) m v (ux)^2 [u(1-x)]$$

$$+ 4k^2 (kq) m (ux)^3 [u(1-x)] - k^2 (kq) m (ux)^3 - 3k^2 (kq) m (ux)^2 [u(1-x)]$$

$$+ k^2 (kq) m (ux)^2 + 2k^2 q^2 q_0 s v (ux) [u(1-x)] - k^2 q^2 q_0 (ux)^2 [u(1-x)]^2$$

$$+ k^2 q^2 q_0 (ux)^2 [u(1-x)] + 2k^2 q^2 q_0 (ux) [u(1-x)]^2 - 2k^2 q^2 q_0 (ux) [u(1-x)]$$

³Additional factor 2 is inserted to take into account both ladder diagrams in Fig.1

$$\begin{aligned}
& -2k^2q^2msv(ux)[u(1-x)] + 2k^2q^2m(ux)^2[u(1-x)]^2 - k^2q^2m(ux)^2[u(1-x)] \\
& \quad - k^2q^2m(ux)[u(1-x)]^2 + 2k^2q^2m(ux)[u(1-x)] - 2k^2q_0^3(ux)^2[u(1-x)]^2 \\
& \quad + 2k^2q_0^3(ux)^2[u(1-x)] - 2k^2q_0^2m(ux)^2[u(1-x)]^2 - 2k^2q_0^2m(ux)^2[u(1-x)] \\
& \quad - 2k^2q_0m^2v(ux) + k^2q_0m^2(ux)^2 - 2k^2q_0m^2(ux) + 2k^2m^3v(ux) - 2k^2m^3(ux)^2 \\
& \quad + k^2m^3(ux) - 4(kq)^2q_0(ux)^2[u(1-x)]^2 + 2(kq)^2q_0(ux)^2[u(1-x)] \\
& \quad + 2(kq)^2q_0(ux)[u(1-x)]^2 - 2(kq)^2m(ux)^2[u(1-x)] - 2(kq)^2m(ux)[u(1-x)]^2 \\
& \quad - 4(kq)q^2q_0s(ux)[u(1-x)]^2 - 6(kq)q^2q_0(ux)[u(1-x)]^3 \\
& \quad + 3(kq)q^2q_0(ux)[u(1-x)]^2 + (kq)q^2q_0[u(1-x)]^3 - (kq)q^2q_0[u(1-x)]^2 \\
& \quad + 4(kq)q^2ms(ux)[u(1-x)]^2 + 4(kq)q^2m(ux)[u(1-x)]^3 \\
& \quad - 3(kq)q^2m(ux)[u(1-x)]^2 - (kq)q^2m[u(1-x)]^3 + (kq)q^2m[u(1-x)]^2 \\
& \quad + 4(kq)q_0^3(ux)[u(1-x)]^3 - 4(kq)q_0^2m(ux)[u(1-x)]^3 + 10(kq)q_0m^2(ux)[u(1-x)] \\
& \quad - (kq)q_0m^2(ux) - (kq)q_0m^2[u(1-x)] - (kq)q_0m^2 - 8(kq)m^3(ux)[u(1-x)] \\
& \quad + (kq)m^3(ux) + (kq)m^3[u(1-x)] + (kq)m^3 + q^4q_0s^2[u(1-x)]^2 - q^4q_0[u(1-x)]^4 \\
& \quad + q^4q_0[u(1-x)]^3 - q^4ms^2[u(1-x)]^2 + q^4m[u(1-x)]^4 - q^4m[u(1-x)]^3 \\
& \quad + 2q^2q_0^3[u(1-x)]^4 - 2q^2q_0^3[u(1-x)]^3 - 2q^2q_0^2m[u(1-x)]^4 + 2q^2q_0^2m[u(1-x)]^3 \\
& \quad - 2q^2q_0m^2s[u(1-x)] + 2q^2q_0m^2[u(1-x)]^2 - q^2q_0m^2[u(1-x)] + 2q^2m^3s[u(1-x)] \\
& \quad - 2q^2m^3[u(1-x)]^2 + q^2m^3[u(1-x)] - 2q_0^3m^2[u(1-x)]^2 - 2q_0^3m^2[u(1-x)] \\
& \quad + 2q_0^2m^3[u(1-x)]^2 + 2q_0^2m^3[u(1-x)].
\end{aligned}$$

Next we are going to calculate the upper loop momentum integral in Fig.1a, b. Due to two photon propagators this integral contains a factor q^4 in the denominator and one may worry about its convergence at small loop momenta. We have checked in the previous section that the light by light scattering tensor supplies at least two powers of loop momentum q in the low-frequency region making the loop integration completely safe. However, the integral over the upper loop momentum q will be taken prior to integration over the Feynman parameters in the light by light scattering loop and in this case the problem of poor low q behavior of the integrand resurrects. In order to make all intermediate integrations infrared safe we temporarily

introduce an intermediate photon mass λ for the photons in the upper loop. All calculations both for the ladder and for the crossed diagrams will be performed with this nonvanishing photon mass. The final expression for the energy shift will admit the limit of vanishing photon mass and will be convergent even in this limit.

Let us combine both electron denominators and photon denominators in the upper loop in Fig.1. To facilitate transformations we introduce new a notation

$$\Delta = -\frac{1}{\gamma}(q^2 + \alpha k^2 - 2\beta(kq) - \gamma m^2), \quad (40)$$

where

$$\begin{aligned} \alpha &= \frac{x(1-ux)}{(1-x)[1-u(1-x)]}, \\ \beta &= \frac{ux}{1-u(1-x)}, \\ \gamma &= \frac{1}{u(1-x)[1-u(1-x)]}. \end{aligned}$$

Then

$$(1-t)[(1-y)(q^2 - \lambda^2) + y(q^2 + 2mq_0)] + t(-\gamma\Delta) = q'^2 - d_\lambda^2, \quad (41)$$

where

$$q' = q + y(1-t)m - \beta tk,$$

$$d_\lambda^2 = -k^2 t(\alpha - \beta^2 t) + m^2[y^2(1-t)^2 + \gamma t] + \lambda^2(1-t)(1-y).$$

Hence,

$$\frac{1}{(q^2 - \lambda^2)^2(q^2 + 2mq_0)\Delta} = -6\gamma \int_0^1 dy(1-y) \int_0^1 dt(1-t)^2 \frac{1}{(q'^2 - d_\lambda^2)^4}, \quad (42)$$

$$\frac{1}{(q^2 - \lambda^2)^2(q^2 + 2mq_0)\Delta^2} = 24\gamma^2 \int_0^1 dy(1-y) \int_0^1 dt t(1-t)^2 \frac{1}{(q'^2 - d_\lambda^2)^5}.$$

Next euclidean rotation is performed followed by calculation of the integral over q_E' ⁴ for the contribution of the ladder diagram to the energy shift in eq.(10).

$$\begin{aligned} \Delta E_l = & \frac{\alpha^2(Z\alpha)^5}{\pi n^3} m \left(\frac{m_r}{m}\right)^3 \frac{24}{\pi^2} \alpha^2(Z\alpha)^5 m \left(\frac{m_r}{m}\right)^3 \int \frac{d^3k}{4\pi k^4} \int_0^1 dx \quad (43) \\ & \int_0^1 du \gamma \int_0^1 dy (1-y) \int_0^1 dt (1-t)^2 \int_0^1 dx \int_0^1 du \int dq'^2 q'^2 \\ & \left\{ (1-u) \frac{F_1^l(q = q' - y(1-t)m + \beta tk)}{(q'^2 + d_\lambda^2)^4} \right. \\ & \left. + 8\gamma u(1-u)t \frac{F_2^l(q = q' - y(1-t)m + \beta tk)}{(q'^2 + d_\lambda^2)^5} \right\}. \end{aligned}$$

Explicit expressions for the numerators in the integrand (after Wick rotation) have the form

$$2F_1^l(q') = -q'^2 m P_1^l + 2k^2 m P_2^l + 2m^3 [y(1-t)]^2 P_3^l, \quad (44)$$

where

$$\begin{aligned} P_1^l = & 18su[u(1-x)][y(1-t)] + 3s(1-x)(1-u)[y(1-t)] \quad (45) \\ & + 6s(1-u)[u(1-x)][y(1-t)] - 18u[u(1-x)]^2[y(1-t)] - 4u[u(1-x)]^2 \\ & + 12u[u(1-x)][y(1-t)] + 6u[u(1-x)] - 3(1-x)(1-u)[u(1-x)][y(1-t)] \\ & - 6(1-u)[u(1-x)]^2[y(1-t)], \end{aligned}$$

$$\begin{aligned} P_2^l = & 6su[u(1-x)][y(1-t)](\beta t)^2 + s(1-x)(1-u)[y(1-t)](\beta t)^2 \quad (46) \\ & + 2s(1-u)[u(1-x)][y(1-t)](\beta t)^2 + 2uvx(1-u)[y(1-t)] + 6uv(ux)[y(1-t)] \\ & - 6u(ux)^2[y(1-t)] - 4u(ux)^2 - 32u(ux)[u(1-x)][y(1-t)](\beta t) \\ & - 8u(ux)[u(1-x)](\beta t) + 2u(ux)[y(1-t)](\beta t) + 4u(ux)[y(1-t)] + 2u(ux)(\beta t) \\ & + 2u(ux) - 10u[u(1-x)]^2[y(1-t)](\beta t)^2 - 4u[u(1-x)]^2(\beta t)^2 \\ & + 2u[u(1-x)][y(1-t)](\beta t)^2 + 2u[u(1-x)][y(1-t)](\beta t) + 2u[u(1-x)](\beta t)^2 \end{aligned}$$

⁴Subscript E is omitted below.

$$\begin{aligned}
& +2u[u(1-x)](\beta t) + 2u[y(1-t)](\beta t) + 2u(\beta t) + vx(1-u)[y(1-t)] \\
& -x(1-u)(ux)[y(1-t)] - 4(1-x)(1-u)(ux)[y(1-t)](\beta t) \\
& -(1-x)(1-u)[u(1-x)][y(1-t)](\beta t)^2 - 2(1-u)(ux)^2[y(1-t)] \\
& -8(1-u)(ux)[u(1-x)][y(1-t)](\beta t) - 2(1-u)[u(1-x)]^2[y(1-t)](\beta t)^2,
\end{aligned}$$

$$\begin{aligned}
P_3^l &= 6su[u(1-x)][y(1-t)] + s(1-x)(1-u)[y(1-t)] \quad (47) \\
& +2s(1-u)[u(1-x)][y(1-t)] - 2u[u(1-x)]^2[y(1-t)] + 4u[u(1-x)]^2 \\
& +6u[u(1-x)][y(1-t)] + 6u[u(1-x)] - (1-x)(1-u)[u(1-x)][y(1-t)] \\
& -2(1-u)[u(1-x)]^2[y(1-t)].
\end{aligned}$$

The second term in the integrand in eq.(43) has the following form

$$2F_2^l = \frac{1}{3}q'^4 m[u(1-x)]^2 T_4^l - q'^2 k^2 m[u(1-x)] T_1^l - q'^2 m^3 [u(1-x)] T_5^l \quad (48)$$

$$+2k^4 m T_2^l + 2k^2 m^3 T_3^l + 2m^5 [u(1-x)][y(1-t)]^2 T_6^l,$$

where

$$\begin{aligned}
T_4^l &= -12s^2[y(1-t)] - 6s^2 + 2[u(1-x)]^2[y(1-t)] \quad (49) \\
& +3[u(1-x)]^2 - 2[u(1-x)][y(1-t)] - 3[u(1-x)],
\end{aligned}$$

$$\begin{aligned}
T_1^l &= -8s^2[u(1-x)][y(1-t)](\beta t)^2 - 6s^2[u(1-x)](\beta t)^2 \quad (50) \\
& -6sv(ux)[y(1-t)] - 4sv(ux) + 16s(ux)[u(1-x)][y(1-t)](\beta t) \\
& +12s(ux)[u(1-x)](\beta t) + 8(ux)^2[u(1-x)][y(1-t)] + 3(ux)^2[u(1-x)] \\
& -7(ux)^2[y(1-t)] - 4(ux)^2 + 18(ux)[u(1-x)]^2[y(1-t)](\beta t) \\
& +10(ux)[u(1-x)]^2(\beta t) - 12(ux)[u(1-x)][y(1-t)](\beta t) - 7(ux)[u(1-x)][y(1-t)] \\
& -9(ux)[u(1-x)](\beta t) - 3(ux)[u(1-x)] + 6(ux)[y(1-t)] + 4(ux) \\
& +5[u(1-x)]^3[y(1-t)](\beta t)^2 + 5[u(1-x)]^3(\beta t)^2 - 5[u(1-x)]^2[y(1-t)](\beta t)^2 \\
& -4[u(1-x)]^2[y(1-t)](\beta t) - 5[u(1-x)]^2(\beta t)^2 - 3[u(1-x)]^2(\beta t) \\
& +4[u(1-x)][y(1-t)](\beta t) + 3[u(1-x)](\beta t),
\end{aligned}$$

$$\begin{aligned}
T_5^l &= -8s^2[u(1-x)][y(1-t)]^3 - 6s^2[u(1-x)][y(1-t)]^2 \quad (51) \\
&+ 6s[y(1-t)] + 4s - 5[u(1-x)]^3[y(1-t)]^3 - 3[u(1-x)]^3[y(1-t)]^2 \\
&+ 5[u(1-x)]^2[y(1-t)]^3 + 3[u(1-x)]^2[y(1-t)]^2 - 3[u(1-x)][y(1-t)] \\
&- 3[u(1-x)] + 6[y(1-t)] + 3,
\end{aligned}$$

$$\begin{aligned}
T_2^l &= -s^2[u(1-x)]^2[y(1-t)](\beta t)^4 - s^2[u(1-x)]^2(\beta t)^4 \quad (52) \\
&- 2sv(ux)[u(1-x)][y(1-t)](\beta t)^2 - 2sv(ux)[u(1-x)](\beta t)^2 \\
&+ 4s(ux)[u(1-x)]^2[y(1-t)](\beta t)^3 + 4s(ux)[u(1-x)]^2(\beta t)^3 \\
&- v^2(ux)^2[y(1-t)] - v^2(ux)^2 + 4v(ux)^2[u(1-x)][y(1-t)](\beta t) \\
&+ 4v(ux)^2[u(1-x)](\beta t) + (ux)^4 + 4(ux)^3[u(1-x)][y(1-t)](\beta t) \\
&+ 4(ux)^3[u(1-x)](\beta t) - (ux)^3[y(1-t)](\beta t) - (ux)^3(\beta t) - (ux)^3 \\
&+ 5(ux)^2[u(1-x)]^2[y(1-t)](\beta t)^2 + 2(ux)^2[u(1-x)]^2(\beta t)^2 \\
&- 3(ux)^2[u(1-x)][y(1-t)](\beta t)^2 - 3(ux)^2[u(1-x)][y(1-t)](\beta t) \\
&- 3(ux)^2[u(1-x)](\beta t)^2 - 3(ux)^2[u(1-x)](\beta t) + (ux)^2[y(1-t)](\beta t) \\
&+ (ux)^2(\beta t) + 6(ux)[u(1-x)]^3[y(1-t)](\beta t)^3 + 4(ux)[u(1-x)]^3(\beta t)^3 \\
&- 3(ux)[u(1-x)]^2[y(1-t)](\beta t)^3 - 4(ux)[u(1-x)]^2[y(1-t)](\beta t)^2 \\
&- 3(ux)[u(1-x)]^2(\beta t)^3 - 3(ux)[u(1-x)]^2(\beta t)^2 + 2(ux)[u(1-x)][y(1-t)](\beta t)^2 \\
&+ 2(ux)[u(1-x)](\beta t)^2 + [u(1-x)]^4[y(1-t)](\beta t)^4 + [u(1-x)]^4(\beta t)^4 \\
&- [u(1-x)]^3[y(1-t)](\beta t)^4 - [u(1-x)]^3[y(1-t)](\beta t)^3 - [u(1-x)]^3(\beta t)^4 \\
&- [u(1-x)]^3(\beta t)^3 + [u(1-x)]^2[y(1-t)](\beta t)^3 + [u(1-x)]^2(\beta t)^3,
\end{aligned}$$

$$\begin{aligned}
T_3^l &= -2s^2[u(1-x)]^2[y(1-t)]^3(\beta t)^2 - 2s^2[u(1-x)]^2[y(1-t)]^2(\beta t)^2 \quad (53) \\
&- 2sv(ux)[u(1-x)][y(1-t)]^3 - 2sv(ux)[u(1-x)][y(1-t)]^2 \\
&+ 4s(ux)[u(1-x)]^2[y(1-t)]^3(\beta t) + 4s(ux)[u(1-x)]^2[y(1-t)]^2(\beta t)
\end{aligned}$$

$$\begin{aligned}
& +2s[u(1-x)][y(1-t)](\beta t)^2 + 2s[u(1-x)](\beta t)^2 + 2v(ux)[y(1-t)] + 2v(ux) \\
& +3(ux)^2[u(1-x)]^2[y(1-t)]^3 - 3(ux)^2[u(1-x)][y(1-t)]^3 \\
& -3(ux)^2[u(1-x)][y(1-t)]^2 - (ux)^2[y(1-t)] - 2(ux)^2 \\
& +2(ux)[u(1-x)]^3[y(1-t)]^3(\beta t) - 3(ux)[u(1-x)]^2[y(1-t)]^3(\beta t) \\
& -2(ux)[u(1-x)]^2[y(1-t)]^3 - 3(ux)[u(1-x)]^2[y(1-t)]^2(\beta t) \\
& -(ux)[u(1-x)]^2[y(1-t)]^2 + 2(ux)[u(1-x)][y(1-t)]^3 + 2(ux)[u(1-x)][y(1-t)]^2 \\
& -10(ux)[u(1-x)][y(1-t)](\beta t) - 8(ux)[u(1-x)](\beta t) + (ux)[y(1-t)](\beta t) \\
& +2(ux)[y(1-t)] + (ux)(\beta t) + (ux) - [u(1-x)]^3[y(1-t)]^3(\beta t) \\
& -[u(1-x)]^3[y(1-t)]^2(\beta t) + [u(1-x)]^2[y(1-t)]^3(\beta t) + [u(1-x)]^2[y(1-t)]^2(\beta t) \\
& -2[u(1-x)]^2[y(1-t)](\beta t)^2 - 2[u(1-x)]^2(\beta t)^2 + [u(1-x)][y(1-t)](\beta t)^2 \\
& +[u(1-x)][y(1-t)](\beta t) + [u(1-x)](\beta t)^2 + [u(1-x)](\beta t) + [y(1-t)](\beta t) + (\beta t),
\end{aligned}$$

$$\begin{aligned}
T_6^l &= -s^2[u(1-x)][y(1-t)]^3 - s^2[u(1-x)][y(1-t)]^2 \quad (54) \\
& +2s[y(1-t)] + 2s - [u(1-x)]^3[y(1-t)]^3 - [u(1-x)]^3[y(1-t)]^2 \\
& +[u(1-x)]^2[y(1-t)]^3 + [u(1-x)]^2[y(1-t)]^2 + 3[y(1-t)] + 3.
\end{aligned}$$

After integration we obtain

$$\begin{aligned}
\Delta E_l &= \frac{\alpha^2(Z\alpha)^5}{\pi n^3} m \left(\frac{m_r}{m}\right)^3 \frac{12}{\pi^2} \int_0^1 dx \int_0^1 du (1-u) \gamma \quad (55) \\
& \int_0^1 dy (1-y) \int_0^1 dt (1-t)^2 \int \frac{d|\mathbf{k}|}{\mathbf{k}^2} \left\{ 4\gamma t \mathbf{k}^4 \frac{muT_2^l}{3d_\lambda^6} \right. \\
& - \mathbf{k}^2 \left[\frac{mP_2^l - 2\gamma tm[u(1-x)]uT_1^l}{3d_\lambda^4} + 4\gamma t \frac{m^3 uT_3^l}{3d_\lambda^6} \right] \\
& - \frac{mP_1^l}{3d_\lambda^2} + \frac{m^3[y(1-t)]^2 P_3^l}{3d_\lambda^4} + \frac{4}{3} \gamma t \frac{m[u(1-x)]^2 uT_4^l}{2d_\lambda^2} \\
& \left. - 4\gamma t \frac{m^3[u(1-x)]uT_5^l}{6d_\lambda^4} + 4\gamma t \frac{m^5[u(1-x)]u[y(1-t)]^2 T_6^l}{3d_\lambda^6} \right\}.
\end{aligned}$$

Integration over $|\mathbf{k}|$ may easily be performed since it is essentially one-dimensional due to spacelike nature of vector k .

It is convenient to introduce the following notation

$$\begin{aligned} d_\lambda^2 &= \mathbf{k}^2 t(\alpha - \beta^2 t) + m^2[y^2(1-t)^2 + \gamma t] + \lambda^2(1-t)(1-y) \\ &\equiv \rho \left\{ \mathbf{k}^2 + \frac{m^2[y^2(1-t)^2 + \gamma t] + \lambda^2(1-t)(1-y)}{\rho} \right\} \\ &= \rho(\mathbf{k}^2 + \omega_\lambda^2), \end{aligned} \quad (56)$$

where

$$\begin{aligned} \rho &\equiv t(\alpha - \beta^2 t), \\ \omega_\lambda^2 &\equiv \frac{m^2[y^2(1-t)^2 + \gamma t] + \lambda^2(1-t)(1-y)}{\rho}. \end{aligned} \quad (57)$$

Integration over $|\mathbf{k}|$ in eq.(55) is impeded by the apparent infrared divergence, which is connected with the apparent constant asymptote of the electron factor.⁵ However, one expects that the leading low frequency term in this asymptote is proportional to \mathbf{k}^2 (if this is not the case the graphs under consideration would produce contributions of even lower order in $Z\alpha$ which is well known to be the wrong conclusion). Surely, only the total electron-line factor should be proportional \mathbf{k}^2 in the low frequency limit and not the part taken into account in eq.(55). These considerations give another opportunity to check the validity of all transformations above. One has to calculate all parts of the electron-line factor and then to calculate its low frequency asymptote. If it turns out that the electron-line factor vanishes as \mathbf{k}^2 when \mathbf{k} is small then there are really no difficulties in the treatment of integrals of the sort contained in eq.(55); one has simply to perform subtraction of the leading low frequency terms in the electron-line factor since these terms cancel in any case in the total electron-line factor. We will check the disappearance of the apparent leading term in the asymptote of the total electron factor in the next Section.

Subtraction of the low frequency part is necessary in the last five terms of the integrand in eq.(55). The numerators of these terms are independent

⁵We call here electron factor the product of the electron factor defined above and the light by light scattering tensor. Normalization of this electron factor differs from the one used in our previous works [2, 3] on the contributions to the Lamb shift; in terms of the old normalization the asymptote we are discussing now is proportional to \mathbf{k}^2 .

of momentum k and subtraction may be easily performed with the help of the identities

$$\begin{aligned}
\frac{1}{d_\lambda^2(\mathbf{k})} - \frac{1}{d_\lambda^2(0)} &= -\frac{\mathbf{k}^2}{\omega_\lambda^2 d_\lambda^2(\mathbf{k})} \equiv -\frac{\mathbf{k}^2}{\rho \omega_\lambda^2 (\mathbf{k}^2 + \omega_\lambda^2)}, \\
\frac{1}{d_\lambda^4(\mathbf{k})} - \frac{1}{d_\lambda^4(0)} &= -\frac{\mathbf{k}^2}{\rho \omega_\lambda^4 d_\lambda^2(\mathbf{k})} - \frac{\mathbf{k}^2}{\omega_\lambda^2 d_\lambda^4(\mathbf{k})} \\
&\equiv -\frac{\mathbf{k}^2}{\rho^2 \omega_\lambda^4 (\mathbf{k}^2 + \omega_\lambda^2)} - \frac{\mathbf{k}^2}{\rho^2 \omega_\lambda^2 (\mathbf{k}^2 + \omega_\lambda^2)^2}, \\
\frac{1}{d_\lambda^6(\mathbf{k})} - \frac{1}{d_\lambda^6(0)} &= -\frac{\mathbf{k}^2}{\rho^2 \omega_\lambda^6 d_\lambda^2(\mathbf{k})} - \frac{\mathbf{k}^2}{\rho \omega_\lambda^4 d_\lambda^4(\mathbf{k})} - \frac{\mathbf{k}^2}{\omega_\lambda^2 d_\lambda^6(\mathbf{k})} \\
&\equiv -\frac{\mathbf{k}^2}{\rho^3 \omega_\lambda^6 (\mathbf{k}^2 + \omega_\lambda^2)} - \frac{\mathbf{k}^2}{\rho^3 \omega_\lambda^4 (\mathbf{k}^2 + \omega_\lambda^2)^2} - \frac{\mathbf{k}^2}{\rho^3 \omega_\lambda^2 (\mathbf{k}^2 + \omega_\lambda^2)^3}.
\end{aligned} \tag{58}$$

Substituting these expressions in eq.(55) we obtain

$$\begin{aligned}
\Delta E_l^{sub} &= \frac{\alpha^2 (Z\alpha)^5}{\pi n^3} m \left(\frac{m_r}{m} \right)^3 \frac{12}{\pi^2} \int_0^1 dx \\
&\int_0^1 du (1-u) \gamma \int_0^1 dy (1-y) \int_0^1 dt (1-t)^2 \int_0^\infty d|\mathbf{k}| \\
&\left\{ \frac{4\gamma t}{3\rho^3 (\mathbf{k}^2 + \omega_\lambda^2)^3} [-m\omega_\lambda^2 u T_2^l - m^3 u T_3^l - \frac{m^5 [u(1-x)] u [y(1-t)]^2 T_6^l}{\omega_\lambda^2}] \right. \\
&+ \frac{1}{3\rho^2 (\mathbf{k}^2 + \omega_\lambda^2)^2} [4\gamma t \frac{m u T_2^l}{\rho} - m P_2^l + 2\gamma t m [u(1-x)] u T_1^l - \frac{m^3 [y(1-t)]^2 P_3^l}{\omega_\lambda^2} \\
&+ 2\gamma t \frac{m^3 [u(1-x)] u T_5^l}{\omega_\lambda^2} - 4\gamma t \frac{m^5 [u(1-x)] u [y(1-t)]^2 T_6^l}{\rho \omega_\lambda^4}] \\
&+ \frac{m}{3\rho \omega^2 (\mathbf{k}^2 + \omega_\lambda^2)} [P_1^l - \frac{m^2 [y(1-t)]^2 P_3^l}{\rho \omega_\lambda^2} - 2\gamma t [u(1-x)]^2 u T_4^l \\
&\left. + 2\gamma t \frac{m^2 [u(1-x)] u T_5^l}{\rho \omega_\lambda^2} - 4\gamma t \frac{m^4 [u(1-x)] u [y(1-t)]^2 T_6^l}{\rho^2 \omega_\lambda^4} \right\}.
\end{aligned} \tag{59}$$

Integrating next over $|\mathbf{k}|$ we obtain

$$\begin{aligned}
\Delta E_l^{sub} &= \frac{\alpha^2(Z\alpha)^5}{\pi n^3} m \left(\frac{m_r}{m}\right)^3 \frac{12}{\pi} \int_0^1 dx \\
&\int_0^1 du(1-u)\gamma \int_0^1 dy(1-y) \int_0^1 dt(1-t)^2 \\
&\left\{ \frac{\gamma t}{4\rho^3\omega_\lambda^5} [-m\omega_\lambda^2 u T_2^l - m^3 u T_3^l - \frac{m^5[u(1-x)]u[y(1-t)]^2 T_6^l}{\omega_\lambda^2}] \right. \\
&+ \frac{1}{12\rho^2\omega_\lambda^3} [4\gamma t \frac{m u T_2^l}{\rho} - m P_2^l + 2\gamma t m [u(1-x)]u T_1^l - \frac{m^3[y(1-t)]^2 P_3^l}{\omega_\lambda^2} \\
&+ 2\gamma t \frac{m^3[u(1-x)]u T_5^l}{\omega_\lambda^2} - 4\gamma t \frac{m^5[u(1-x)]u[y(1-t)]^2 T_6^l}{\rho\omega_\lambda^4}] \\
&+ \frac{m}{6\rho\omega_\lambda^3} [P_1^l - \frac{m^2[y(1-t)]^2 P_3^l}{\rho\omega_\lambda^2} - 2\gamma t [u(1-x)]^2 u T_4^l \\
&\left. + 2\gamma t \frac{m^2[u(1-x)]u T_5^l}{\rho\omega_\lambda^2} - 4\gamma t \frac{m^4[u(1-x)]u[y(1-t)]^2 T_6^l}{\rho^2\omega_\lambda^4}] \right\}.
\end{aligned} \tag{60}$$

4.2 CROSSED DIAGRAM

Consideration of the crossed diagram contribution to the Lamb shift follows the same lines as in the case of the ladder diagram. First we multiply the respective part of the light by light scattering tensor by the numerator of the electron factor in eq.(3) and obtain

$$\begin{aligned}
L_c &= A_{\alpha\beta} \tilde{C}_{\alpha\beta 00}^{sub} \\
&= - \int_0^1 dx \int_0^1 du \int_0^1 dz \left\{ \frac{u}{\Delta_z} 2F_1^c + \frac{uz^2}{\Delta_z^2} 2F_2^c \right\},
\end{aligned} \tag{61}$$

where

$$\begin{aligned}
F_1^c &= -8k^2 q_0 a^2 z^2 + 8k^2 q_0 a z^2 + 6k^2 q_0 v^2 z^3 (1-z) \\
&+ 4k^2 q_0 v z^3 (ux) + 2k^2 q_0 v (ux) + 2k^2 q_0 z^4 (ux)^2 - 4k^2 q_0 z^4 (ux) + 2k^2 q_0 z^4 \\
&- 2k^2 q_0 z^3 (1-z)(ux)^2 + 4k^2 q_0 z^3 (1-z)(ux) - 2k^2 q_0 z^3 (1-z) - k^2 q_0 z^2 \\
&- 2k^2 q_0 z (ux)^2 + 4k^2 q_0 z (ux) - 2k^2 q_0 z + 2k^2 q_0 (1-z)(ux)^2 - 4k^2 q_0 (1-z)(ux)
\end{aligned} \tag{62}$$

$$\begin{aligned}
& +2k^2q_0(1-z) - 2k^2a^2mz^2 + 2k^2amz^2 - k^2mz^2 - 12(kq)q_0abz^2 + 6(kq)q_0az^2 \\
& +6(kq)q_0bz^2 + 12(kq)q_0svz^3(1-z) + 4(kq)q_0z^4(ux)[u(1-x)] - 4(kq)q_0z^4(ux) \\
& \quad -4(kq)q_0z^4[u(1-x)] + 4(kq)q_0z^4 - 4(kq)q_0z^3(1-z)(ux)[u(1-x)] \\
& \quad +4(kq)q_0z^3(1-z)(ux) + 4(kq)q_0z^3(1-z)[u(1-x)] - 4(kq)q_0z^3(1-z) \\
& -8(kq)q_0z^3(ux)[u(1-x)] - 3(kq)q_0z^2 - 4(kq)q_0z(ux)[u(1-x)] + 4(kq)q_0z(ux) \\
& \quad +4(kq)q_0z[u(1-x)] - 4(kq)q_0z + 4(kq)q_0(1-z)(ux)[u(1-x)] \\
& \quad -4(kq)q_0(1-z)(ux) - 4(kq)q_0(1-z)[u(1-x)] + 4(kq)q_0(1-z) \\
& -4(kq)q_0(ux)[u(1-x)] - 4(kq)abmz^2 + 2(kq)amz^2 + 2(kq)bmz^2 - (kq)mz^2 \\
& \quad -10q^2q_0b^2z^2 + 10q^2q_0bz^2 + 6q^2q_0s^2z^3(1-z) + 4q^2q_0sz^3[u(1-x)] \\
& \quad +2q^2q_0s[u(1-x)] + 2q^2q_0z^4[u(1-x)]^2 - 4q^2q_0z^4[u(1-x)] + 2q^2q_0z^4 \\
& -2q^2q_0z^3(1-z)[u(1-x)]^2 + 4q^2q_0z^3(1-z)[u(1-x)] - 2q^2q_0z^3(1-z) - 2q^2q_0z^2 \\
& \quad -2q^2q_0z[u(1-x)]^2 + 4q^2q_0z[u(1-x)] - 2q^2q_0z + 2q^2q_0(1-z)[u(1-x)]^2 \\
& -4q^2q_0(1-z)[u(1-x)] + 2q^2q_0(1-z) - 2q^2b^2mz^2 + 2q^2bmz^2 - q^2mz^2 + 4q_0^3b^2z^2 \\
& \quad -4q_0^3bz^2 + 2q_0^3z^2 - 4q_0^2b^2mz^2 + 4q_0^2bmz^2 - 2q_0^2mz^2,
\end{aligned}$$

$$\begin{aligned}
F_2^c &= 2k^4q_0a^4 - 4k^4q_0a^3 + 2k^4q_0a^2 - 2k^4q_0v^4z^2(1-z)^2 \quad (63) \\
& -4k^4q_0v^3z^2(1-z)(ux) - 2k^4q_0v^2z^2(ux)^2 + k^4a^4m - 2k^4a^3m + k^4a^2m \\
& -k^4mv^4z^2(1-z)^2 - 2k^4mv^3z^2(1-z)(ux) - k^4mv^2z^2(ux)^2 + 4k^2(kq)q_0a^3b \\
& -2k^2(kq)q_0a^3 - 6k^2(kq)q_0a^2b + 3k^2(kq)q_0a^2 + 2k^2(kq)q_0ab - k^2(kq)q_0a \\
& \quad -8k^2(kq)q_0sv^3z^2(1-z)^2 - 8k^2(kq)q_0sv^2z^2(1-z)(ux) \\
& \quad +8k^2(kq)q_0v^2z^2(1-z)(ux)[u(1-x)] + 8k^2(kq)q_0vz^2(ux)^2[u(1-x)] \\
& +4k^2(kq)a^3bm - 2k^2(kq)a^3m - 6k^2(kq)a^2bm + 3k^2(kq)a^2m + 2k^2(kq)abm \\
& \quad -k^2(kq)am - 4k^2(kq)msv^3z^2(1-z)^2 - 4k^2(kq)msv^2z^2(1-z)(ux) \\
& \quad +4k^2(kq)mv^2z^2(1-z)(ux)[u(1-x)] + 4k^2(kq)mvz^2(ux)^2[u(1-x)] \\
& +4k^2q^2q_0a^2b^2 - 4k^2q^2q_0a^2b + 2k^2q^2q_0a^2 - 4k^2q^2q_0ab^2 + 4k^2q^2q_0ab - 2k^2q^2q_0a
\end{aligned}$$

$$\begin{aligned}
& +k^2q^2q_0b^2 - k^2q^2q_0b - 4k^2q^2q_0s^2v^2z^2(1-z)^2 - 4k^2q^2q_0s^2vz^2(1-z)(ux) \\
& -4k^2q^2q_0sv^2z^2(1-z)[u(1-x)] - 4k^2q^2q_0svz^2(ux)[u(1-x)] + 2k^2q^2a^2b^2m \\
& -2k^2q^2a^2bm + k^2q^2a^2m - 2k^2q^2ab^2m + 2k^2q^2abm - k^2q^2am + k^2q^2b^2m - k^2q^2bm \\
& -2k^2q^2ms^2v^2z^2(1-z)^2 - 2k^2q^2ms^2vz^2(1-z)(ux) - 2k^2q^2msv^2z^2(1-z)[u(1-x)] \\
& -2k^2q^2msvz^2(ux)[u(1-x)] - 4k^2q_0^3a^2b^2 + 4k^2q_0^3a^2b - 2k^2q_0^3a^2 + 4k^2q_0^3ab^2 \\
& -4k^2q_0^3ab + 2k^2q_0^3a + 2k^2q_0^2a^2m - 2k^2q_0^2am - 4k^2q_0a^2m^2 + 4k^2q_0am^2 \\
& +2k^2q_0m^2v^2z^2(1-z) + 2k^2q_0m^2v^2z(1-z)^2 + 2k^2q_0m^2v^2z(1-z) \\
& +2k^2q_0m^2vz^2(ux) + 2k^2q_0m^2vz(1-z)(ux) + 2k^2q_0m^2vz(ux) - k^2q_0m^2 \\
& -2k^2a^2m^3 + 2k^2am^3 + k^2m^3v^2z^2(1-z) + k^2m^3v^2z(1-z)^2 + k^2m^3v^2z(1-z) \\
& +k^2m^3vz^2(ux) + k^2m^3vz(1-z)(ux) + k^2m^3vz(ux) - k^2m^3 \\
& -8(kq)^2q_0s^2v^2z^2(1-z)^2 + 16(kq)^2q_0svz^2(1-z)(ux)[u(1-x)] \\
& -8(kq)^2q_0z^2(ux)^2[u(1-x)]^2 + 4(kq)^2a^2b^2m - 4(kq)^2a^2bm - 4(kq)^2ab^2m \\
& +4(kq)^2abm - 4(kq)^2ms^2v^2z^2(1-z)^2 + 8(kq)^2msvz^2(1-z)(ux)[u(1-x)] \\
& -4(kq)^2mz^2(ux)^2[u(1-x)]^2 + 4(kq)q^2q_0ab^3 - 6(kq)q^2q_0ab^2 + 2(kq)q^2q_0ab \\
& -2(kq)q^2q_0b^3 + 3(kq)q^2q_0b^2 - (kq)q^2q_0b - 8(kq)q^2q_0s^3vz^2(1-z)^2 \\
& -8(kq)q^2q_0s^2vz^2(1-z)[u(1-x)] + 8(kq)q^2q_0s^2z^2(1-z)(ux)[u(1-x)] \\
& +8(kq)q^2q_0sz^2(ux)[u(1-x)]^2 + 4(kq)q^2ab^3m - 6(kq)q^2ab^2m + 2(kq)q^2abm \\
& -2(kq)q^2b^3m + 3(kq)q^2b^2m - (kq)q^2bm - 4(kq)q^2ms^3vz^2(1-z)^2 \\
& -4(kq)q^2ms^2vz^2(1-z)[u(1-x)] + 4(kq)q^2ms^2z^2(1-z)(ux)[u(1-x)] \\
& +4(kq)q^2msz^2(ux)[u(1-x)]^2 - 4(kq)q_0abm^2 + 2(kq)q_0am^2 + 2(kq)q_0bm^2 \\
& +4(kq)q_0m^2svz^2(1-z) + 4(kq)q_0m^2svz(1-z)^2 + 4(kq)q_0m^2svz(1-z) \\
& -4(kq)q_0m^2z^2(ux)[u(1-x)] - 4(kq)q_0m^2z(1-z)(ux)[u(1-x)] \\
& -4(kq)q_0m^2z(ux)[u(1-x)] - (kq)q_0m^2 - 4(kq)abm^3 + 2(kq)am^3 + 2(kq)bm^3 \\
& +2(kq)m^3svz^2(1-z) + 2(kq)m^3svz(1-z)^2 + 2(kq)m^3svz(1-z)
\end{aligned}$$

$$\begin{aligned}
& -2(kq)m^3z^2(ux)[u(1-x)] - 2(kq)m^3z(1-z)(ux)[u(1-x)] \\
& -2(kq)m^3z(ux)[u(1-x)] - (kq)m^3 + 2q^4q_0b^4 - 4q^4q_0b^3 + 2q^4q_0b^2 \\
& -2q^4q_0s^4z^2(1-z)^2 - 4q^4q_0s^3z^2(1-z)[u(1-x)] - 2q^4q_0s^2z^2[u(1-x)]^2 + q^4b^4m \\
& -2q^4b^3m + q^4b^2m - q^4ms^4z^2(1-z)^2 - 2q^4ms^3z^2(1-z)[u(1-x)] \\
& -q^4ms^2z^2[u(1-x)]^2 - 4q^2q_0b^2m^2 + 4q^2q_0bm^2 + 2q^2q_0m^2s^2z^2(1-z) \\
& + 2q^2q_0m^2s^2z(1-z)^2 + 2q^2q_0m^2s^2z(1-z) + 2q^2q_0m^2sz^2[u(1-x)] \\
& + 2q^2q_0m^2sz(1-z)[u(1-x)] + 2q^2q_0m^2sz[u(1-x)] - 2q^2q_0m^2 - 2q^2b^2m^3 + 2q^2bm^3 \\
& + q^2m^3s^2z^2(1-z) + q^2m^3s^2z(1-z)^2 + q^2m^3s^2z(1-z) + q^2m^3sz^2[u(1-x)] \\
& + q^2m^3sz(1-z)[u(1-x)] + q^2m^3sz[u(1-x)] - q^2m^3 + 2q_0^3m^2 - 2q_0^2m^3.
\end{aligned}$$

Next we go to calculation of the upper photon loop in Fig.1c. To facilitate transformations we introduce new notation which resembles the one used in the previous section

$$\begin{aligned}
\Delta_z & \equiv m^2 - z\{k^2(1-ux)[1-z(1-ux)] + q^2[1-u(1-x)][1-z(1-u(1-x))]\} \quad (64) \\
& + 2kq[1-u-z(1-ux)(1-u(1-x))]\} \\
& = -\frac{1}{\gamma_z}[q^2 + \alpha_z k^2 - 2\beta_z(kq) - \gamma_z m^2],
\end{aligned}$$

where

$$\begin{aligned}
\alpha_z & = \frac{(1-ux)[1-z(1-ux)]}{[1-u(1-x)][1-z(1-u(1-x))]}, \\
\beta_z & = -\frac{[1-u-z(1-ux)(1-u(1-x))]}{[1-u(1-x)][1-z(1-u(1-x))]}, \\
\gamma_z & = \frac{1}{z[1-u(1-x)][1-z(1-u(1-x))]} .
\end{aligned}$$

Note that

$$\begin{aligned}
\alpha_z(z=1) & = \alpha, \\
\beta_z(z=1) & = \beta,
\end{aligned} \tag{65}$$

$$\gamma_z(z=1) = \gamma.$$

As was discussed in detail in the case of the ladder diagram we have to introduce small photon mass λ for the photons in the upper loop to make all intermediate integrations infrared safe.

Then

$$(1-t)[(1-y)(q^2 - \lambda^2) + y(q^2 + 2mq_0)] + t(-\gamma_z \Delta_z) = q''^2 - d_{\lambda z}^2, \quad (66)$$

where

$$q'' = q + y(1-t)m - \beta_z tk,$$

$$d_{\lambda z}^2 = -k^2 t(\alpha_z - \beta_z^2 t) + m^2[y^2(1-t)^2 + \gamma_z t] + \lambda^2(1-t)(1-y).$$

Hence,

$$\begin{aligned} \frac{1}{(q^2 - \lambda^2)^2(q^2 + 2mq_0)\Delta_z} &= -6\gamma_z \int_0^1 dy(1-y) \int_0^1 dt(1-t)^2 \frac{1}{(q''^2 - d_{\lambda z}^2)^4}, \\ \frac{1}{(q^2 - \lambda^2)^2(q^2 + 2mq_0)\Delta_z^2} &= 24\gamma_z^2 \int_0^1 dy(1-y) \int_0^1 dt(1-t)^2 \frac{1}{(q''^2 - d_{\lambda z}^2)^5}. \end{aligned} \quad (67)$$

Next we perform euclidean rotation and calculate the integral over q_E'' ⁶ for the contribution of the crossed diagram to the energy shift in eq.(10).

$$\begin{aligned} \Delta E_c &= \frac{\alpha^2(Z\alpha)^5}{\pi n^3} m \left(\frac{m_r}{m}\right)^3 \frac{12}{\pi^2} \int \frac{d^3 k}{4\pi k^4} \int_0^1 dx \\ &\int_0^1 duu \int_0^1 dz \gamma_z \int_0^1 dy(1-y) \int_0^1 dt(1-t)^2 \int dq''^2 q''^2 \\ &\left\{ \frac{2F_1^c(q = q'' - y(1-t)m + \beta_z tk)}{(q''^2 + d_{\lambda z}^2)^4} + 4\gamma_z t \frac{2z^2 F_2^c(q = q'' - y(1-t)m + \beta_z tk)}{(q''^2 + d_{\lambda z}^2)^5} \right\}. \end{aligned} \quad (68)$$

Explicit expressions for the numerators (after Wick rotation) in the integrand have the form

$$2F_1^c = -3q''^2 m P_1^c + 2k^2 m P_2^c + 2m^3[y(1-t)]^2 P_3^c, \quad (69)$$

where

⁶Subscript E is omitted below.

$$\begin{aligned}
P_1^c &= 8b^2z^2[y(1-t)] - 2b^2z^2 - 8bz^2[y(1-t)] + 2bz^2 \quad (70) \\
&- 6s^2z^3(1-z)[y(1-t)] - 4sz^3[u(1-x)][y(1-t)] - 2s[u(1-x)][y(1-t)] \\
&- 2z^4[u(1-x)]^2[y(1-t)] + 4z^4[u(1-x)][y(1-t)] - 2z^4[y(1-t)] \\
&+ 2z^3(1-z)[u(1-x)]^2[y(1-t)] - 4z^3(1-z)[u(1-x)][y(1-t)] \\
&+ 2z^3(1-z)[y(1-t)] + z^2[y(1-t)] - z^2 + 2z[u(1-x)]^2[y(1-t)] \\
&- 4z[u(1-x)][y(1-t)] + 2z[y(1-t)] - 2(1-z)[u(1-x)]^2[y(1-t)] \\
&+ 4(1-z)[u(1-x)][y(1-t)] - 2(1-z)[y(1-t)],
\end{aligned}$$

$$\begin{aligned}
P_2^c &= 8a^2z^2[y(1-t)] - 2a^2z^2 + 12abz^2[y(1-t)](\beta_z t) \quad (71) \\
&- 4abz^2(\beta_z t) - 6az^2[y(1-t)](\beta_z t) - 8az^2[y(1-t)] + 2az^2(\beta_z t) + 2az^2 \\
&+ 10b^2z^2[y(1-t)](\beta_z t)^2 - 2b^2z^2(\beta_z t)^2 - 10bz^2[y(1-t)](\beta_z t)^2 - 6bz^2[y(1-t)](\beta_z t) \\
&+ 2bz^2(\beta_z t)^2 + 2bz^2(\beta_z t) - 6s^2z^3(1-z)[y(1-t)](\beta_z t)^2 \\
&- 12svz^3(1-z)[y(1-t)](\beta_z t) - 4sz^3[u(1-x)][y(1-t)](\beta_z t)^2 \\
&- 2s[u(1-x)][y(1-t)](\beta_z t)^2 - 6v^2z^3(1-z)[y(1-t)] - 4vz^3(ux)[y(1-t)] \\
&- 2v(ux)[y(1-t)] - 2z^4(ux)^2[y(1-t)] - 4z^4(ux)[u(1-x)][y(1-t)](\beta_z t) \\
&+ 4z^4(ux)[y(1-t)](\beta_z t) + 4z^4(ux)[y(1-t)] - 2z^4[u(1-x)]^2[y(1-t)](\beta_z t)^2 \\
&+ 4z^4[u(1-x)][y(1-t)](\beta_z t)^2 + 4z^4[u(1-x)][y(1-t)](\beta_z t) - 2z^4[y(1-t)](\beta_z t)^2 \\
&- 4z^4[y(1-t)](\beta_z t) - 2z^4[y(1-t)] + 2z^3(1-z)(ux)^2[y(1-t)] \\
&+ 4z^3(1-z)(ux)[u(1-x)][y(1-t)](\beta_z t) - 4z^3(1-z)(ux)[y(1-t)](\beta_z t) \\
&- 4z^3(1-z)(ux)[y(1-t)] + 2z^3(1-z)[u(1-x)]^2[y(1-t)](\beta_z t)^2 \\
&- 4z^3(1-z)[u(1-x)][y(1-t)](\beta_z t)^2 - 4z^3(1-z)[u(1-x)][y(1-t)](\beta_z t) \\
&+ 2z^3(1-z)[y(1-t)](\beta_z t)^2 + 4z^3(1-z)[y(1-t)](\beta_z t) + 2z^3(1-z)[y(1-t)] \\
&+ 8z^3(ux)[u(1-x)][y(1-t)](\beta_z t) + 2z^2[y(1-t)](\beta_z t)^2 + 3z^2[y(1-t)](\beta_z t) \\
&+ z^2[y(1-t)] - z^2(\beta_z t)^2 - z^2(\beta_z t) - z^2 + 2z(ux)^2[y(1-t)]
\end{aligned}$$

$$\begin{aligned}
& +4z(ux)[u(1-x)][y(1-t)](\beta_z t) - 4z(ux)[y(1-t)](\beta_z t) - 4z(ux)[y(1-t)] \\
& \quad + 2z[u(1-x)]^2[y(1-t)](\beta_z t)^2 - 4z[u(1-x)][y(1-t)](\beta_z t)^2 \\
& - 4z[u(1-x)][y(1-t)](\beta_z t) + 2z[y(1-t)](\beta_z t)^2 + 4z[y(1-t)](\beta_z t) + 2z[y(1-t)] \\
& \quad - 2(1-z)(ux)^2[y(1-t)] - 4(1-z)(ux)[u(1-x)][y(1-t)](\beta_z t) \\
& \quad + 4(1-z)(ux)[y(1-t)](\beta_z t) + 4(1-z)(ux)[y(1-t)] \\
& \quad - 2(1-z)[u(1-x)]^2[y(1-t)](\beta_z t)^2 + 4(1-z)[u(1-x)][y(1-t)](\beta_z t)^2 \\
& + 4(1-z)[u(1-x)][y(1-t)](\beta_z t) - 2(1-z)[y(1-t)](\beta_z t)^2 - 4(1-z)[y(1-t)](\beta_z t) \\
& \quad - 2(1-z)[y(1-t)] + 4(ux)[u(1-x)][y(1-t)](\beta_z t),
\end{aligned}$$

$$\begin{aligned}
P_3^c &= 6b^2 z^2 [y(1-t)] - 6b^2 z^2 - 6bz^2 [y(1-t)] + 6bz^2 \quad (72) \\
& - 6s^2 z^3 (1-z)[y(1-t)] - 4sz^3 [u(1-x)][y(1-t)] - 2s[u(1-x)][y(1-t)] \\
& \quad - 2z^4 [u(1-x)]^2 [y(1-t)] + 4z^4 [u(1-x)][y(1-t)] - 2z^4 [y(1-t)] \\
& + 2z^3 (1-z)[u(1-x)]^2 [y(1-t)] - 4z^3 (1-z)[u(1-x)][y(1-t)] + 2z^3 (1-z)[y(1-t)] \\
& \quad - 3z^2 + 2z[u(1-x)]^2 [y(1-t)] - 4z[u(1-x)][y(1-t)] + 2z[y(1-t)] \\
& - 2(1-z)[u(1-x)]^2 [y(1-t)] + 4(1-z)[u(1-x)][y(1-t)] - 2(1-z)[y(1-t)],
\end{aligned}$$

Second term in the integrand in eq.(68) has the form

$$\begin{aligned}
2F_2^c &= 2q''^4 m T_4^c - q''^2 k^2 m T_1^c - q''^2 m^3 T_5^c \quad (73) \\
& + 2k^4 m T_2^c + 2k^2 m^3 T_3^c + 2m^5 [y(1-t)]^2 T_6^c,
\end{aligned}$$

where ⁷

$$\begin{aligned}
T_4^c &= -4b^4 [y(1-t)] + b^4 + 8b^3 [y(1-t)] - 2b^3 \quad (74) \\
& - 4b^2 [y(1-t)] + b^2 + 4s^4 z^2 (1-z)^2 [y(1-t)] - s^4 z^2 (1-z)^2 \\
& + 8s^3 z^2 (1-z)[u(1-x)][y(1-t)] - 2s^3 z^2 (1-z)[u(1-x)] \\
& + 4s^2 z^2 [u(1-x)]^2 [y(1-t)] - s^2 z^2 [u(1-x)]^2,
\end{aligned}$$

⁷It is not difficult to check that despite its cumbersome appearance function T_4^c vanishes identically: $T_4^c \equiv 0$.

$$\begin{aligned}
T_1^c = & -6a^2b^2[y(1-t)] + 6a^2b^2 + 6a^2b[y(1-t)] - 6a^2b \quad (75) \\
& -3a^2[y(1-t)] + 3a^2 - 16ab^3[y(1-t)](\beta_z t) + 12ab^3(\beta_z t) + 24ab^2[y(1-t)](\beta_z t) \\
& + 6ab^2[y(1-t)] - 18ab^2(\beta_z t) - 6ab^2 - 8ab[y(1-t)](\beta_z t) - 6ab[y(1-t)] \\
& + 6ab(\beta_z t) + 6ab + 3a[y(1-t)] - 3a - 16b^4[y(1-t)](\beta_z t)^2 + 6b^4(\beta_z t)^2 \\
& + 32b^3[y(1-t)](\beta_z t)^2 + 8b^3[y(1-t)](\beta_z t) - 12b^3(\beta_z t)^2 - 6b^3(\beta_z t) \\
& - 16b^2[y(1-t)](\beta_z t)^2 - 12b^2[y(1-t)](\beta_z t) - 3b^2[y(1-t)] + 6b^2(\beta_z t)^2 + 9b^2(\beta_z t) \\
& + 2b^2 + 4b[y(1-t)](\beta_z t) + 3b[y(1-t)] - 3b(\beta_z t) - 2b + 16s^4z^2(1-z)^2[y(1-t)](\beta_z t)^2 \\
& - 6s^4z^2(1-z)^2(\beta_z t)^2 + 32s^3vz^2(1-z)^2[y(1-t)](\beta_z t) - 12s^3vz^2(1-z)^2(\beta_z t) \\
& + 32s^3z^2(1-z)[u(1-x)][y(1-t)](\beta_z t)^2 - 12s^3z^2(1-z)[u(1-x)](\beta_z t)^2 \\
& + 16s^2v^2z^2(1-z)^2[y(1-t)] - 6s^2v^2z^2(1-z)^2 + 12s^2vz^2(1-z)(ux)[y(1-t)] \\
& - 4s^2vz^2(1-z)(ux) + 32s^2vz^2(1-z)[u(1-x)][y(1-t)](\beta_z t) \\
& - 12s^2vz^2(1-z)[u(1-x)](\beta_z t) - 32s^2z^2(1-z)(ux)[u(1-x)][y(1-t)](\beta_z t) \\
& + 12s^2z^2(1-z)(ux)[u(1-x)](\beta_z t) + 16s^2z^2[u(1-x)]^2[y(1-t)](\beta_z t)^2 \\
& - 6s^2z^2[u(1-x)]^2(\beta_z t)^2 + 12svz^2(1-z)[u(1-x)][y(1-t)] \\
& - 4svz^2(1-z)[u(1-x)] - 8svz^2(1-z)(ux)[u(1-x)][y(1-t)] \\
& + 4svz^2(1-z)(ux)[u(1-x)] + 12svz^2(ux)[u(1-x)][y(1-t)] \\
& - 4svz^2(ux)[u(1-x)] - 32sz^2(ux)[u(1-x)]^2[y(1-t)](\beta_z t) \\
& + 12sz^2(ux)[u(1-x)]^2(\beta_z t) + 4z^2(ux)^2[u(1-x)]^2[y(1-t)] \\
& - 2z^2(ux)^2[u(1-x)]^2,
\end{aligned}$$

$$\begin{aligned}
T_5^c = & -16b^4[y(1-t)]^3 + 6b^4[y(1-t)]^2 + 32b^3[y(1-t)]^3 \quad (76) \\
& - 12b^3[y(1-t)]^2 - 16b^2[y(1-t)]^3 + 6b^2[y(1-t)]^2 + 12b^2[y(1-t)] - 4b^2 \\
& - 12b[y(1-t)] + 4b + 16s^4z^2(1-z)^2[y(1-t)]^3 - 6s^4z^2(1-z)^2[y(1-t)]^2 \\
& + 32s^3z^2(1-z)[u(1-x)][y(1-t)]^3 - 12s^3z^2(1-z)[u(1-x)][y(1-t)]^2
\end{aligned}$$

$$\begin{aligned}
& -6s^2z^2(1-z)[y(1-t)] + 2s^2z^2(1-z) + 16s^2z^2[u(1-x)]^2[y(1-t)]^3 \\
& -6s^2z^2[u(1-x)]^2[y(1-t)]^2 - 6s^2z(1-z)^2[y(1-t)] + 2s^2z(1-z)^2 \\
& -6s^2z(1-z)[y(1-t)] + 2s^2z(1-z) - 6sz^2[u(1-x)][y(1-t)] + 2sz^2[u(1-x)] \\
& -6sz(1-z)[u(1-x)][y(1-t)] + 2sz(1-z)[u(1-x)] - 6sz[u(1-x)][y(1-t)] \\
& + 2sz[u(1-x)] + 3[y(1-t)] - 3,
\end{aligned}$$

$$\begin{aligned}
T_2^c = & -2a^4[y(1-t)] + a^4 - 4a^3b[y(1-t)](\beta_z t) + 4a^3b(\beta_z t) \quad (77) \\
& + 2a^3[y(1-t)](\beta_z t) + 4a^3[y(1-t)] - 2a^3(\beta_z t) - 2a^3 - 4a^2b^2[y(1-t)](\beta_z t)^2 \\
& + 6a^2b^2(\beta_z t)^2 + 4a^2b[y(1-t)](\beta_z t)^2 + 6a^2b[y(1-t)](\beta_z t) - 6a^2b(\beta_z t)^2 - 6a^2b(\beta_z t) \\
& - 2a^2[y(1-t)](\beta_z t)^2 - 3a^2[y(1-t)](\beta_z t) - 2a^2[y(1-t)] + a^2(\beta_z t)^2 + 3a^2(\beta_z t) \\
& + a^2 - 4ab^3[y(1-t)](\beta_z t)^3 + 4ab^3(\beta_z t)^3 + 6ab^2[y(1-t)](\beta_z t)^3 + 4ab^2[y(1-t)](\beta_z t)^2 \\
& - 6ab^2(\beta_z t)^3 - 6ab^2(\beta_z t)^2 - 2ab[y(1-t)](\beta_z t)^3 - 4ab[y(1-t)](\beta_z t)^2 \\
& - 2ab[y(1-t)](\beta_z t) + 2ab(\beta_z t)^3 + 6ab(\beta_z t)^2 + 2ab(\beta_z t) + 2a[y(1-t)](\beta_z t)^2 \\
& + a[y(1-t)](\beta_z t) - a(\beta_z t)^2 - a(\beta_z t) - 2b^4[y(1-t)](\beta_z t)^4 + b^4(\beta_z t)^4 \\
& + 4b^3[y(1-t)](\beta_z t)^4 + 2b^3[y(1-t)](\beta_z t)^3 - 2b^3(\beta_z t)^4 - 2b^3(\beta_z t)^3 \\
& - 2b^2[y(1-t)](\beta_z t)^4 - 3b^2[y(1-t)](\beta_z t)^3 - b^2[y(1-t)](\beta_z t)^2 + b^2(\beta_z t)^4 + 3b^2(\beta_z t)^3 \\
& + b^2(\beta_z t)^2 + b[y(1-t)](\beta_z t)^3 + b[y(1-t)](\beta_z t)^2 - b(\beta_z t)^3 - b(\beta_z t)^2 \\
& + 2s^4z^2(1-z)^2[y(1-t)](\beta_z t)^4 - s^4z^2(1-z)^2(\beta_z t)^4 + 8s^3vz^2(1-z)^2[y(1-t)](\beta_z t)^3 \\
& - 4s^3vz^2(1-z)^2(\beta_z t)^3 + 4s^3z^2(1-z)[u(1-x)][y(1-t)](\beta_z t)^4 \\
& - 2s^3z^2(1-z)[u(1-x)](\beta_z t)^4 + 12s^2vz^2(1-z)^2[y(1-t)](\beta_z t)^2 \\
& - 6s^2vz^2(1-z)^2(\beta_z t)^2 + 4s^2vz^2(1-z)(ux)[y(1-t)](\beta_z t)^2 \\
& - 2s^2vz^2(1-z)(ux)(\beta_z t)^2 + 8s^2vz^2(1-z)[u(1-x)][y(1-t)](\beta_z t)^3 \\
& - 4s^2vz^2(1-z)[u(1-x)](\beta_z t)^3 - 8s^2z^2(1-z)(ux)[u(1-x)][y(1-t)](\beta_z t)^3 \\
& + 4s^2z^2(1-z)(ux)[u(1-x)](\beta_z t)^3 + 2s^2z^2[u(1-x)]^2[y(1-t)](\beta_z t)^4 \\
& - s^2z^2[u(1-x)]^2(\beta_z t)^4 + 8sv^3z^2(1-z)^2[y(1-t)](\beta_z t) - 4sv^3z^2(1-z)^2(\beta_z t)
\end{aligned}$$

$$\begin{aligned}
& +8sv^2z^2(1-z)(ux)[y(1-t)](\beta_zt) - 4sv^2z^2(1-z)(ux)(\beta_zt) \\
& +4sv^2z^2(1-z)[u(1-x)][y(1-t)](\beta_zt)^2 - 2sv^2z^2(1-z)[u(1-x)](\beta_zt)^2 \\
& -16svz^2(1-z)(ux)[u(1-x)][y(1-t)](\beta_zt)^2 + 8svz^2(1-z)(ux)[u(1-x)](\beta_zt)^2 \\
& +4svz^2(ux)[u(1-x)][y(1-t)](\beta_zt)^2 - 2svz^2(ux)[u(1-x)](\beta_zt)^2 \\
& -8sz^2(ux)[u(1-x)]^2[y(1-t)](\beta_zt)^3 + 4sz^2(ux)[u(1-x)]^2(\beta_zt)^3 \\
& +2v^4z^2(1-z)^2[y(1-t)] - v^4z^2(1-z)^2 + 4v^3z^2(1-z)(ux)[y(1-t)] \\
& -2v^3z^2(1-z)(ux) - 8v^2z^2(1-z)(ux)[u(1-x)][y(1-t)](\beta_zt) \\
& +4v^2z^2(1-z)(ux)[u(1-x)](\beta_zt) + 2v^2z^2(ux)^2[y(1-t)] - v^2z^2(ux)^2 \\
& -8vz^2(ux)^2[u(1-x)][y(1-t)](\beta_zt) + 4vz^2(ux)^2[u(1-x)](\beta_zt) \\
& +8z^2(ux)^2[u(1-x)]^2[y(1-t)](\beta_zt)^2 - 4z^2(ux)^2[u(1-x)]^2(\beta_zt)^2,
\end{aligned}$$

$$\begin{aligned}
T_3^c &= 2a^2b^2[y(1-t)]^2 - 2a^2b[y(1-t)]^2 + 3a^2[y(1-t)]^2 \quad (78) \\
& +4a^2[y(1-t)] - 2a^2 - 4ab^3[y(1-t)]^3(\beta_zt) + 4ab^3[y(1-t)]^2(\beta_zt) \\
& +6ab^2[y(1-t)]^3(\beta_zt) - 6ab^2[y(1-t)]^2(\beta_zt) - 2ab^2[y(1-t)]^2 - 2ab[y(1-t)]^3(\beta_zt) \\
& +2ab[y(1-t)]^2(\beta_zt) + 2ab[y(1-t)]^2 + 4ab[y(1-t)](\beta_zt) - 4ab(\beta_zt) - 3a[y(1-t)]^2 \\
& -2a[y(1-t)](\beta_zt) - 4a[y(1-t)] + 2a(\beta_zt) + 2a - 4b^4[y(1-t)]^3(\beta_zt)^2 \\
& +2b^4[y(1-t)]^2(\beta_zt)^2 + 8b^3[y(1-t)]^3(\beta_zt)^2 + 2b^3[y(1-t)]^3(\beta_zt) \\
& -4b^3[y(1-t)]^2(\beta_zt)^2 - 2b^3[y(1-t)]^2(\beta_zt) - 4b^2[y(1-t)]^3(\beta_zt)^2 \\
& -3b^2[y(1-t)]^3(\beta_zt) - b^2[y(1-t)]^3 + 2b^2[y(1-t)]^2(\beta_zt)^2 + 3b^2[y(1-t)]^2(\beta_zt) \\
& +b^2[y(1-t)]^2 + 4b^2[y(1-t)](\beta_zt)^2 - 2b^2(\beta_zt)^2 + b[y(1-t)]^3(\beta_zt) + b[y(1-t)]^3 \\
& -b[y(1-t)]^2(\beta_zt) - b[y(1-t)]^2 - 4b[y(1-t)](\beta_zt)^2 - 2b[y(1-t)](\beta_zt) + 2b(\beta_zt)^2 \\
& +2b(\beta_zt) + 4s^4z^2(1-z)^2[y(1-t)]^3(\beta_zt)^2 - 2s^4z^2(1-z)^2[y(1-t)]^2(\beta_zt)^2 \\
& +8s^3vz^2(1-z)^2[y(1-t)]^3(\beta_zt) - 4s^3vz^2(1-z)^2[y(1-t)]^2(\beta_zt) \\
& +8s^3z^2(1-z)[u(1-x)][y(1-t)]^3(\beta_zt)^2 - 4s^3z^2(1-z)[u(1-x)][y(1-t)]^2(\beta_zt)^2 \\
& +4s^2v^2z^2(1-z)^2[y(1-t)]^3 - 2s^2v^2z^2(1-z)^2[y(1-t)]^2
\end{aligned}$$

$$\begin{aligned}
& +4s^2vz^2(1-z)(ux)[y(1-t)]^3 - 2s^2vz^2(1-z)(ux)[y(1-t)]^2 \\
& +8s^2vz^2(1-z)[u(1-x)][y(1-t)]^3(\beta_z t) - 4s^2vz^2(1-z)[u(1-x)][y(1-t)]^2(\beta_z t) \\
& \quad - 8s^2z^2(1-z)(ux)[u(1-x)][y(1-t)]^3(\beta_z t) \\
& +4s^2z^2(1-z)(ux)[u(1-x)][y(1-t)]^2(\beta_z t) - 2s^2z^2(1-z)[y(1-t)](\beta_z t)^2 \\
& \quad + s^2z^2(1-z)(\beta_z t)^2 + 4s^2z^2[u(1-x)]^2[y(1-t)]^3(\beta_z t)^2 \\
& -2s^2z^2[u(1-x)]^2[y(1-t)]^2(\beta_z t)^2 - 2s^2z(1-z)^2[y(1-t)](\beta_z t)^2 + s^2z(1-z)^2(\beta_z t)^2 \\
& -2s^2z(1-z)[y(1-t)](\beta_z t)^2 + s^2z(1-z)(\beta_z t)^2 + 4sv^2z^2(1-z)[u(1-x)][y(1-t)]^3 \\
& -2sv^2z^2(1-z)[u(1-x)][y(1-t)]^2 - 4svz^2(1-z)[y(1-t)](\beta_z t) + 2svz^2(1-z)(\beta_z t) \\
& \quad + 4svz^2(ux)[u(1-x)][y(1-t)]^3 - 2svz^2(ux)[u(1-x)][y(1-t)]^2 \\
& -4svz(1-z)^2[y(1-t)](\beta_z t) + 2svz(1-z)^2(\beta_z t) - 4svz(1-z)[y(1-t)](\beta_z t) \\
& \quad + 2svz(1-z)(\beta_z t) - 8sz^2(ux)[u(1-x)]^2[y(1-t)]^3(\beta_z t) \\
& +4sz^2(ux)[u(1-x)]^2[y(1-t)]^2(\beta_z t) - 2sz^2[u(1-x)][y(1-t)](\beta_z t)^2 \\
& \quad + s^2z[u(1-x)](\beta_z t)^2 - 2sz(1-z)[u(1-x)][y(1-t)](\beta_z t)^2 \\
& \quad + sz(1-z)[u(1-x)](\beta_z t)^2 - 2sz[u(1-x)][y(1-t)](\beta_z t)^2 \\
& \quad + sz[u(1-x)](\beta_z t)^2 - 2v^2z^2(1-z)[y(1-t)] + v^2z^2(1-z) \\
& -2v^2z(1-z)^2[y(1-t)] + v^2z(1-z)^2 - 2v^2z(1-z)[y(1-t)] + v^2z(1-z) \\
& -2vz^2(ux)[y(1-t)] + vz^2(ux) - 2vz(1-z)(ux)[y(1-t)] + vz(1-z)(ux) \\
& \quad - 2vz(ux)[y(1-t)] + vz(ux) + 4z^2(ux)[u(1-x)][y(1-t)](\beta_z t) \\
& -2z^2(ux)[u(1-x)](\beta_z t) + 4z(1-z)(ux)[u(1-x)][y(1-t)](\beta_z t) \\
& \quad - 2z(1-z)(ux)[u(1-x)](\beta_z t) + 4z(ux)[u(1-x)][y(1-t)](\beta_z t) \\
& -2z(ux)[u(1-x)](\beta_z t) + 2[y(1-t)](\beta_z t)^2 + [y(1-t)](\beta_z t) + [y(1-t)] \\
& \quad - (\beta_z t)^2 - (\beta_z t) - 1,
\end{aligned}$$

$$\begin{aligned}
T_6^c &= -2b^4[y(1-t)]^3 + b^4[y(1-t)]^2 + 4b^3[y(1-t)]^3 \quad (79) \\
& -2b^3[y(1-t)]^2 - 2b^2[y(1-t)]^3 + b^2[y(1-t)]^2 + 4b^2[y(1-t)] - 2b^2 - 4b[y(1-t)]
\end{aligned}$$

$$\begin{aligned}
& +2b + 2s^4 z^2 (1-z)^2 [y(1-t)]^3 - s^4 z^2 (1-z)^2 [y(1-t)]^2 \\
& + 4s^3 z^2 (1-z) [u(1-x)] [y(1-t)]^3 - 2s^3 z^2 (1-z) [u(1-x)] [y(1-t)]^2 \\
& - 2s^2 z^2 (1-z) [y(1-t)] + s^2 z^2 (1-z) + 2s^2 z^2 [u(1-x)]^2 [y(1-t)]^3 \\
& - s^2 z^2 [u(1-x)]^2 [y(1-t)]^2 - 2s^2 z (1-z)^2 [y(1-t)] + s^2 z (1-z)^2 \\
& - 2s^2 z (1-z) [y(1-t)] + s^2 z (1-z) - 2s z^2 [u(1-x)] [y(1-t)] + s z^2 [u(1-x)] \\
& - 2s z (1-z) [u(1-x)] [y(1-t)] + s z (1-z) [u(1-x)] - 2s z [u(1-x)] [y(1-t)] \\
& + s z [u(1-x)] - 3.
\end{aligned}$$

After integration we obtain

$$\begin{aligned}
\Delta E_c = & \frac{\alpha^2 (Z\alpha)^5}{\pi n^3} m \left(\frac{m_r}{m}\right)^3 \frac{12}{\pi^2} \int_0^1 dx \int_0^1 du u \int_0^1 dz \gamma_z \quad (80) \\
& \int_0^1 dy (1-y) \int_0^1 dt (1-t)^2 \int \frac{d|\mathbf{k}|}{\mathbf{k}^2} \left\{ 4\gamma_z t \mathbf{k}^4 \frac{m z^2 T_2^c}{6d_{\lambda z}^6} \right. \\
& - \mathbf{k}^2 \left[\frac{m P_2^c - \gamma_z t m z^2 T_1^c}{3d_{\lambda z}^4} + 4\gamma_z t \frac{m^3 z^2 T_3^c}{6d_{\lambda z}^6} \right] \\
& - \frac{3m P_1^c}{3d_{\lambda z}^2} + \frac{m^3 [y(1-t)]^2 P_3^c}{3d_{\lambda z}^4} + 4\gamma_z t \frac{m z^2 T_4^c}{2d_{\lambda z}^2} \\
& \left. - 2\gamma_z t \frac{m^3 z^2 T_5^c}{6d_{\lambda z}^4} + 4\gamma_z t \frac{m^5 z^2 [y(1-t)]^2 T_6^c}{6d_{\lambda z}^6} \right\}.
\end{aligned}$$

Integration over $|\mathbf{k}|$ is no more difficult than in the ladder case. We introduce notation which resembles the one used in the ladder case

$$\begin{aligned}
d_{\lambda z}^2 = & \mathbf{k}^2 t (\alpha_z - \beta_z^2 t) + m^2 [y^2 (1-t)^2 + \gamma_z t] \quad (81) \\
\equiv & \rho_z \left\{ \mathbf{k}^2 + \frac{m^2 [y^2 (1-t)^2 + \gamma_z t] + \lambda^2 (1-t)(1-y)}{\rho_z} \right\} \\
= & \rho_z (\mathbf{k}^2 + \omega_{\lambda z}^2),
\end{aligned}$$

where

$$\begin{aligned}
\rho_z \equiv & t (\alpha_z - \beta_z^2 t), \quad (82) \\
\omega_{\lambda z}^2 \equiv & \frac{m^2 [y^2 (1-t)^2 + \gamma_z t] + \lambda^2 (1-t)(1-y)}{\rho_z}.
\end{aligned}$$

Once again as in the ladder case we encounter the problem of apparent divergence of the integral for the energy shift. As was already explained it is necessary to check that the total electron-line factor vanishes as \mathbf{k}^2 in the low frequency region. It is an easy task to do. We simply omit integration over \mathbf{k} in eq.(55) and eq.(80) and put \mathbf{k} to be equal to zero in the integrand. Then we obtain

$$\begin{aligned}
\Delta\epsilon_{test} &= \Delta\epsilon_l + \Delta\epsilon_c \tag{83} \\
&= \int_0^1 dx \int_0^1 du \int_0^1 dy (1-y) \int_0^1 dt (1-t)^2 \\
&\quad \left\{ \gamma \left\{ -\frac{(1-u)P_1^l}{3[y^2(1-t)^2 + \gamma t]} + \frac{[y(1-t)]^2(1-u)P_3^l}{3[y^2(1-t)^2 + \gamma t]^2} + 4\gamma t \frac{[u(1-x)]^2 u(1-u)T_4^l}{6[y^2(1-t)^2 + \gamma t]} \right. \right. \\
&\quad \left. \left. - 4\gamma t \frac{[u(1-x)]u(1-u)T_5^l}{6[y^2(1-t)^2 + \gamma t]^2} + 4\gamma t \frac{[u(1-x)]u[y(1-t)]^2(1-u)T_6^l}{3[y^2(1-t)^2 + \gamma t]^3} \right\} \right. \\
&\quad + \int_0^1 dz \gamma_z \left\{ -\frac{3uP_1^c}{3[y^2(1-t)^2 + \gamma_z t]} + \frac{u[y(1-t)]^2 P_3^c}{3[y^2(1-t)^2 + \gamma_z t]^2} \right. \\
&\quad + 4\gamma_z t \frac{z^2 u T_4^c}{2[y^2(1-t)^2 + \gamma_z t]} - 2\gamma_z t \frac{z^2 u T_5^c}{6[y^2(1-t)^2 + \gamma_z t]^2} \\
&\quad \left. \left. + 4\gamma_z t \frac{z^2 u [y(1-t)]^2 T_6^c}{6[y^2(1-t)^2 + \gamma_z t]^3} \right\} \right\}.
\end{aligned}$$

We calculated this integral numerically and it turned out to be equal zero as expected. Then there are really no difficulties in the treatment of the integrals contained in eq.(80); one has simply to perform subtraction of the leading low frequency terms in the electron-line factor since these terms cancel in any case in the total electron-line factor.

Subtraction of the low frequency part is necessary in the last five terms of the integrand in eq.(80). The numerators of these terms are independent of momentum k and subtraction may be easily performed with the help of the identities

$$\begin{aligned}
\frac{1}{d_{\lambda z}^2(\mathbf{k})} - \frac{1}{d_{\lambda z}^2(0)} &= -\frac{\mathbf{k}^2}{\omega_{\lambda z}^2 d_{\lambda z}^2(\mathbf{k})} \equiv -\frac{\mathbf{k}^2}{\rho_z \omega_{\lambda z}^2 (\mathbf{k}^2 + \omega_{\lambda z}^2)}, \tag{84} \\
\frac{1}{d_{\lambda z}^4(\mathbf{k})} - \frac{1}{d_{\lambda z}^4(0)} &= -\frac{\mathbf{k}^2}{\rho_z \omega_{\lambda z}^4 d_{\lambda z}^2(\mathbf{k})} - \frac{\mathbf{k}^2}{\omega_{\lambda z}^2 d_z^4(\mathbf{k})}
\end{aligned}$$

$$\begin{aligned}
&\equiv -\frac{\mathbf{k}^2}{\rho_z^2 \omega_{\lambda z}^4 (\mathbf{k}^2 + \omega_{\lambda z}^2)} - \frac{\mathbf{k}^2}{\rho_z^2 \omega_{\lambda z}^2 (\mathbf{k}^2 + \omega_{\lambda z}^2)^2}, \\
\frac{1}{d_{\lambda z}^6(\mathbf{k})} - \frac{1}{d_{\lambda z}^6(0)} &= -\frac{\mathbf{k}^2}{\rho_z^2 \omega_{\lambda z}^6 d_{\lambda z}^2(\mathbf{k})} - \frac{\mathbf{k}^2}{\rho_z \omega_{\lambda z}^4 d_{\lambda z}^4(\mathbf{k})} - \frac{\mathbf{k}^2}{\omega_z^2 d_{\lambda z}^6(\mathbf{k})} \\
&\equiv -\frac{\mathbf{k}^2}{\rho_z^3 \omega_{\lambda z}^6 (\mathbf{k}^2 + \omega_{\lambda z}^2)} - \frac{\mathbf{k}^2}{\rho_z^3 \omega_{\lambda z}^4 (\mathbf{k}^2 + \omega_{\lambda z}^2)^2} - \frac{\mathbf{k}^2}{\rho_z^3 \omega_{\lambda z}^2 (\mathbf{k}^2 + \omega_{\lambda z}^2)^3}.
\end{aligned}$$

Substituting these subtractions in eq.(80) and performing the momentum integration we obtain

$$\begin{aligned}
\Delta E_c^{sub} &= \frac{\alpha^2 (Z\alpha)^5}{\pi n^3} m \left(\frac{m_r}{m} \right)^3 \frac{12}{\pi} \int_0^1 dx \quad (85) \\
&\int_0^1 du u \int_0^1 dz \gamma_z \int_0^1 dy (1-y) \int_0^1 dt (1-t)^2 \\
&\left\{ \frac{\gamma_z t}{8 \rho_z^3 \omega_{\lambda z}^5} [-m \omega_{\lambda z}^2 z^2 T_2^c - m^3 z^2 T_3^c - \frac{m^5 z^2 [y(1-t)]^2 T_6^c}{\omega_{\lambda z}^2}] \right. \\
&+ \frac{1}{12 \rho_z^2 \omega_{\lambda z}^3} [2 \gamma_z t \frac{m z^2 T_2^c}{\rho_z} - m P_2^c + \gamma_z t m z^2 T_1^c - \frac{m^3 [y(1-t)]^2 P_3^c}{\omega_{\lambda z}^2} \\
&\quad \left. + \gamma_z t \frac{m^3 z^2 T_5^c}{\omega_{\lambda z}^2} - 2 \gamma_z t \frac{m^5 z^2 [y(1-t)]^2 T_6^c}{\rho_z \omega_{\lambda z}^4}] \right. \\
&+ \frac{m}{6 \rho_z \omega_{\lambda z}^3} [3 P_1^c - \frac{m^2 [y(1-t)]^2 P_3^c}{\rho_z \omega_{\lambda z}^2} - 6 \gamma_z t z^2 T_4^c \\
&\quad \left. + \gamma_z t \frac{m^2 z^2 T_5^c}{\rho_z \omega_{\lambda z}^2} - 2 \gamma_z t \frac{m^4 z^2 [y(1-t)]^2 T_6^c}{\rho_z^2 \omega_{\lambda z}^4}] \right\}.
\end{aligned}$$

5 TOTAL CONTRIBUTION TO THE LAMB SHIFT AND DISCUSSION OF RESULTS

It is straightforward to calculate numerically the total contribution to the S -level Lamb shift of order $\alpha^2 (Z\alpha)^5 m$ induced by the light by light scattering insertion. This contribution is given by the sum of the expressions in eq.(60) and in eq.(85). The only subtlety is that these expressions contain an auxiliary infrared regularizing parameter λ . As we have mentioned

above the final formula for the energy shift should be given in the limit of the vanishing photon mass and should produce an unambiguous value for the contribution to the Lamb shift. We have checked numerically that the result of integration is independent of this small intermediate photon mass as it is varied from $\lambda^2 = 10^{-4}$ to $\lambda^2 = 10^{-8}$. This limited range of values for the auxiliary photon mass is defined by the limited accuracy and productivity of the computer and may be widened at the expense of significantly more computer time. Our final result is

$$\Delta E = -0.122(2) \frac{\alpha^2(Z\alpha)^5}{\pi n^3} \left(\frac{m_r}{m}\right)^3 m. \quad (86)$$

One may readily obtain more digits in the expression above if necessary.

In this work, initiated in [1], we have finished calculation of all corrections to the Lamb shift of order $\alpha^2(Z\alpha)^5 m$ induced by the diagrams with closed electron loops. Our results are presented in the Table.⁸

Corrections to the Lamb shift
induced by the diagrams with closed electron loops

ΔE	$\times \frac{\alpha^2(Z\alpha)^5}{\pi n^3} \left(\frac{m_r}{m}\right)^3 m$	2S (kHz)	1S (kHz)
$a^{[1]}$	-0.061	-0.329	-2.63
$b^{[1]}$	0.508	2.747	21.98
$c^{[2]}$	0.611	3.305	26.44
$d^{[3]}$	-0.073	-0.394	-3.15
<i>e-this work</i>	-0.122(2)	-0.660	-5.28
Total	0.863(2)	4.67(1)	37.3(1)

The theoretical contributions to the Lamb shift presented in the Table are clearly necessary for comparison of theory with recent experiments on measurement of the $2S_{1/2} - 2P_{1/2}$ splitting in hydrogen [6, 7]

$$\Delta\nu = 1057\,845\,(9)\,\text{kHz},$$

⁸While this paper was in preparation a work [5] appeared which contains recalculation and confirmation of the results obtained in [1, 2, 3] as well as the first calculation of the light by light contribution in a framework completely different from the one used in this paper and in [1, 2, 3]. The result for the light by light contribution presented above is in agreement with the respective result in [5].

$$\Delta\nu = 1057\,851.4\,(1.9)\,\text{kHz}, \quad (87)$$

and on measurement of the $1S$ Lamb shift [8]

$$\Delta\nu_{1S} = 8\,172.82\,(11)\,\text{MHz}. \quad (88)$$

Work on the calculation of the still unknown corrections of order $\alpha^2(Z\alpha)^5m$ produced by the diagrams with two radiative photon insertions in the electron line is in the finishing stage now by at least two groups⁹.

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⁹K. Pachucki, submitted for publication; M. I. Eides, S. G. Karshenboim and V. A. Shelyuto [9, 10] and paper in preparation.

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Figure Caption

Fig.1. Gauge invariant set of diagrams with light by light scattering insertions in the external Coulomb lines.

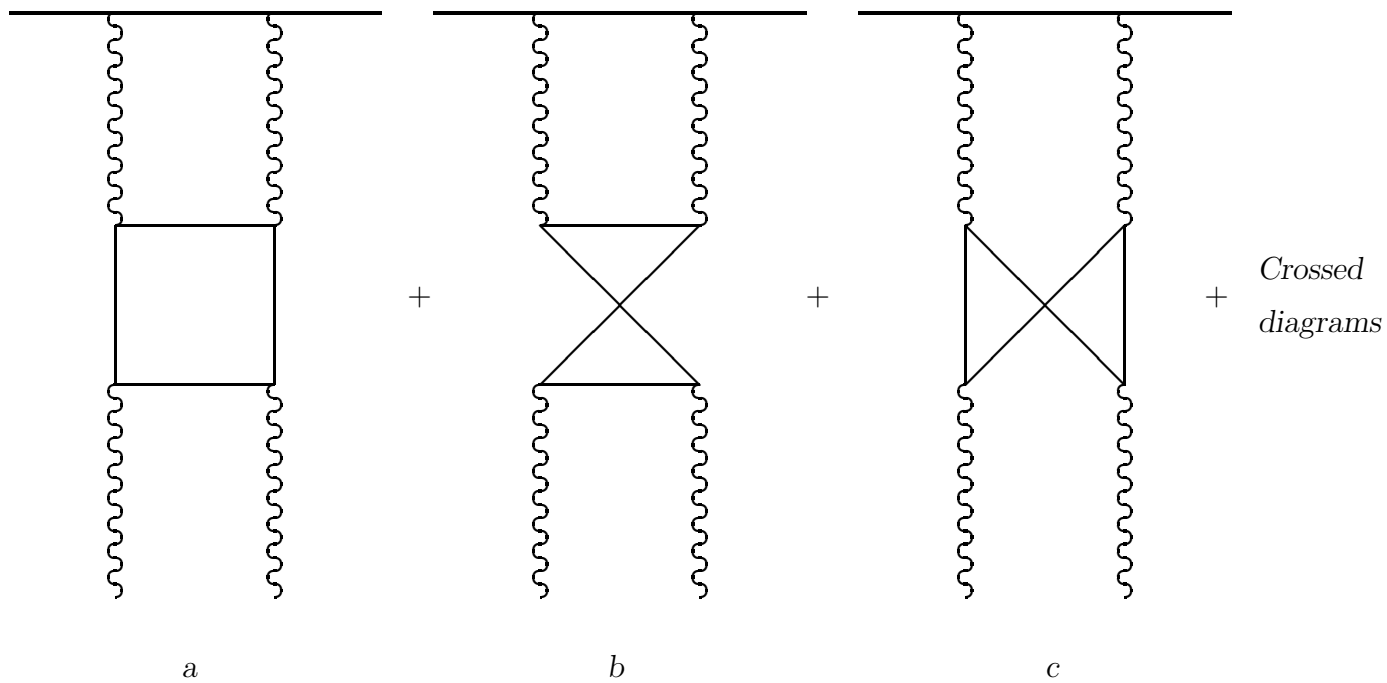


Fig.1